# Reflexions on Teaching Methods of Proof to College Students and Students' Difficulties

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*Methods of proof teached are conditional proof, biconditional proof, induction proof, proofs containing quantifiers « it exists » and « for all », contradiction method, contrapositive method, and uniqueness proofs. One of the challenges is to build the course in respect of constructivism theories. What mathematical elements does the teacher have to consider in preparing students to understand these methods? In what order, will he present to the class these types of proof? What are the contents chosen? Students' difficulties are encountered for each method of proof they are learning. Making the difference between hypothesis and conclusion, using hypothesis of induction, constructing a mathematical object, finding a contradiction and making the difference between uniqueness methods are examples of these difficulties.*

## INTRODUCTION

The aim of this paper is to share an experience of teaching methods of proof to college students. The first part concerns the teaching of methods. To organize the teaching, the teacher has to choose methods, find the best order of presenting them and prepare students to face main difficulties. The second part is the description of students' difficulties in analysing, completing, or writing a proof.

## TEACHING METHODS OF PROOF

### Preliminaries to learn methods of proof

The first part of teaching has the aim of preparing the students to learn methods of proof. First, the student has to learn to make the difference between an hypothesis and a conclusion. He has to check the key-word « then ». Usually, the hypothesis is written before this key-word and the conclusion is written after it. Even if it seems easy to do, some students may still confuse the hypothesis and the conclusion in equations.

While students figure out this difference, we introduce truth tables of conjunction, disjunction, negation, tautology, contradiction, conditional, and biconditional propositions. For example, using a truth table or the algebra of propositions, a student has to show that a proposition is logically equivalent to another, or that it is a tautology, or that it is a contradiction. To support our objective of making the difference between an hypothesis and a conclusion, we also differentiate conditional, converse, inverse, and contrapositive propositions in presenting the truth tables.

In the same section of the course, we give examples of lemmas, corollaries, and axioms. The content is mainly algebraic and geometrical. We also try to prove a proposition using another proposition or using a definition (this is difficult for students). To put emphasis on this objective, we have been inspired by Solow (2005) who introduced the idea of « key question » and « answer to the key question ». It is a pedagogical approach where the student asks himself (and writes) « How to show this conclusion? ». He answers himself to that question (and writes the answer) by using a definition, a proposition, or any property he knows. The teacher provides to students a list of definitions to be used during the class and exams. Each definition or proposition leads the student to another conclusion that he has to show.

To prepare students to understand the methods of proof, we present them two quantifiers « it exists » and « for all », as they appear often in definitions. They analyse the statement after the quantifier by identifying the object, the property, and the consequence (in the same way that Solow does). To prepare students to the contradiction and the contrapositive methods, they make exercises on the negation of statements containing these quantifiers.

**Methods of proof and the order of presenting them**

We can group together induction method and the method of proof containing the quantifier « for all ». Exercises linked to the last method are related to set theory, functions and calculus (because we use « for all »). Exercises linked to induction method are mainly related to numbers theory and calculus.

The method of proof containing the quantifier « it exists » presents obstacles for the students'understanding. On the other hand, these exercises are easier because they are a revision of calculus problems they have done during their first year of college.

Some books present the induction method before the other ones. Is it easier for the students? We introduce the method containing the quantifier « it exists » or the « constructive method » before the first group because students are already familiar with the content of calculus.

The contrapositive and the contradiction methods follow the first two groups, because it is necessary that student make the negation of statements containing the quantifiers « it exists » or « for all ». Some handbooks introduce these methods without presenting methods of proof related to the quantifiers.

In the last section of the course, we introduce the uniqueness methods of proof. In the indirect method, the students use the contradiction method. In the indirect and direct uniqueness methods, the « constructive method » is necessary.

This order is chosen in respect of constructivist theories.

**Choosen content**

In this section, for each type of proof, we will present examples of exercises that students have to do. They have to complete a proof with missing statements or explanations, correct errors, or write the proof.

Exercises on Rolle's theorem and on the mean value theorem, where it is necessary to find a real value, are examples related to the « constructive method ». At the same time, it is an occasion to make a review of the concepts of continuity and derivation. The « constructive method » is also present when the students have to prove that an algebraic expression is even, odd, or rational.

Examples of exercises related to proofs containing the quantifier « for all » or « method of choice » are the ones where students have to show that « for all real numbers of a set, a function or a sequence is increasing or decreasing ». Students use the notion of derivation.

Proving the sum of a series, or an inequality, or that an expression is divisible by a number are examples of proofs relates to the induction method. The knowledge of algebra is useful here.

There are also exercises where the quantifiers « for all » and « it exists » are nested. Two cases may happen. In the first one, the quantifier « it exists » comes before the quantifier « for all » in the conclusion. In the second case, it is the opposite. In calculus, there are many exercises that are linked to these situations. The example « Let a sequence, there exists a real number k, with for all n > 1, then the sequence converge toward k » represents a proposition of the first situation and « Let a function f(x), for all real numbers on an interval, there exists a real number in another interval such as f(x) = y » represents a proposition of the second situation.

The content of the contrapositive and the contradiction methods is various. We can show that a number is irrational, that an expression is not divisible by a number, that a number is not the solution of an equation, that a function is one-to-one, and that a number is even or odd. Usually, these methods are applied, when the conclusion is easy to negate.

Finally, we find in linear algebra many examples of uniqueness methods as showing that a linear combination of linearly independant vectors is unique and that the solution of a system of linear equations is unique.  We can find examples in calculus to demonstrate that if a limit exists, it is unique and  in number theory to show that the « gcd  » of two integers is unique.

The first step in choosing the content of proof is to make a review of calculus and linear algebra theories. Usually, in the calculus courses, we don't have time to do demonstrations because we are very busy teaching algebra. The second step is to select the content to be presented to the students (as new notions of graph theory and numbers theory) that would promote the methods of proof seen in the course.

## STUDENTS' DIFFICULTIES

This section will be divided in two parts : the first part covers a description of students' difficulties concerning the preliminaries and second part, the description of their difficulties on the methods of proof.

### Preliminaries and students' difficulties

Usually, when students read a proposition statement, they are able to identify the hypothesis and the conclusion. But, if there is an equation or an inequation to be showned in the conclusion, some students begin the proof with the two parts simultaneously (side left and side right) of the expression rather than starting with one side (of the equation or inequation) and then trying to get, with properties, definitions, or theorems to the other side.

The use of a definition or a theorem to prove a proposition remains difficult to many students. During the first weeks of the course, students have to develop a reflex of consulting their definitions' list and they have to reformulate the conclusion with respect of the definition found. Using a theorem to prove a proposition is harder, and even after showing many examples, some students still seem not to understand. Their main problems are in the formulation of the conclusion (to prove) as the hypothesis of the theorem, and the acceptance that the proof (of the proposition) is finished when the hypothesis is proved.

Generally, truth tables are easy for students and they like to validate a proposition with them or with the algebra of propositions. The analysis of quantifiers doesn't seem to cause major difficulties to students.

### Methods of proof and students' difficulties

On the other hand, students have difficulties to accept, in the « constructive method » of proof, that the object is showed at the beginning of the proof and that the proof is the verification that the object leads to the conclusion. To turn this problem, students write the proof backward starting from the conclusion and find the object or the solution as they did in calculus (for Lagrange or Rolle theorems). Sometimes, a rough copy helps to find the object of the proof containing the quantifier « it exists ».

The other type of proof containing a quantifier « for all » generates difficulties because the exercises are related to sets theory and students are not familiar with it. Even, if the elements of sets are expressed as algebraic expressions or interval, students seem uncomfortable with the sets and the « choice method ». Many students are unable to use the hypothesis of induction in this method. As this method necessitates algebraic abilities, they have difficulties to substitute the algebraic expression of the hypothesis of induction (for n) in the algebraic expression to prove (for n+1). In another way, they have also difficulties to modify the expression to prove (for n+1) to be able to use the hypothesis of induction (for n).

In the contrapositive and contradiction methods, since students have done exercises in negating a statement, it seems that they were able to write the right hypothesis of these methods. The main difficulties observed are related to the contradiction method. The way students expressed the contradiction is often clumsy, because they don’t have the exact vocabulary to explain it or because they have difficulties with algebra. The contrapositive method seems easier for them, because they already know what to prove, the negation of the hypothesis. As they know what they have to demonstrate they may use an argumentation not always adequate, even a tricky or doubting one.

In the exams, students didn’t have to choose by themselves between contrapositive or contradiction methods, and between indirect or direct uniqueness methods to do a demonstration : the method was suggested in the question. We note that the direct method was easier to apply than the indirect method. Proving with the indirect method necessitates using the contradiction method. As the contents of the proofs were relatively new, students had more difficulties with the mathematical contents than with the form of these two methods.

Students had also difficulties to complete a proof with missing details. They were unable to identify at what stage of the demonstration, there was a missing detail. They had a tendency to rewrite the demonstration without completing it.

## CONCLUSION

Many of these difficulties that we just described suggest an important problem : many students have difficulty in making links between two mathematical concepts. At the end of the course, some students develop this ability. Explanatory elements of their difficulties could be linked to the meaning they give to mathematical terms and symbols. Returning to « previous mathematical knowledge », borrowing from « neighbouring mathematical knowledge » and interference of « common sense » with definition of certain mathematical terms could also give explanations to the students' difficulties (Maurice, 2000). As this paper is a discussion of a second experience of teaching methods of proof, our reflexions are tentative. To better understand students' difficulties it would be necessary to elaborate a serious study of the teaching method of proofs.

## References

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