# SECONDARY TEACHERS' PROFESSIONAL DEVELOPMENT THROUGH THE EXPLORATION OF SCHOOL MATHEMATICS

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**Abstract**: In this paper, I report on a year-long professional development project with secondary mathematics teachers, structured around monthly in-service group sessions. These sessions were focused on the exploration of school mathematics concepts, that is, the very mathematics teachers teach in their everyday practices. Grounded in the research literature on teachers' knowledge of mathematics and an excerpt of data from one of the sessions, I offer some implications for the professional development of secondary mathematics teachers.

In this paper, I report on a professional development initiative set up for secondary mathematics teachers. Six teachers (Carole, Carl, Eric, Lana, Linda & Gina) participated in the project with an intention to revitalize and improve their mathematics teaching practices. The teacher generally felt their mathematical knowledge was too focused on procedural knowledge and that this orientation had an important impact on their teaching practices. Some of them expressed the following: "Why is it that we are not able to solve by reasoning? […] It is because we have not been educated to reason in mathematics. Me, I did copy, paste, repeat, and let's go … and I had 95% in mathematics!" (Carole) or "I never understood why it worked… When students ask me why, I simply say that this is how it is! (Lana)".

This situation with regards to secondary mathematics teachers is not new, as important critiques have been raised about the mathematical preparation of schoolteachers. Indeed, studies point to difficulties that secondary teachers have experienced with aspects of the mathematics they teach. Studies by Ball (1990) and Bryan (1999) have illustrated that while the secondary mathematics teachers they studied made few if any mistakes in their usage of mathematical procedures, they experienced significant difficulties in providing sound meaning and explanations for the mathematical rationales underlying these same procedures. Other studies have highlighted difficulties of a different order concerning their unfamiliarity with the meaning of concepts and solving processes (e.g., definitions, conjectures, relationships within concepts). Even (1993) and Hitt-Espinosa (1998) have observed that many teachers possessed an "old" definition of a function as a continuous graph, preventing them from recognizing or accepting alternative drawings as representing a function and leading them to transform or treat discrete functions as continuous. Also, Schmidt and Bednarz (1997) and Van Dooren, Verschaffel and Onghena (2003) have reported on secondary teachers ease with algebraic solutions, but difficulties in appreciating arithmetical procedures as valid solutions to algebraic problems.

Despite criticism that these types of studies present a "deficit model" of teacher knowledge, and moreover not being generalizable to all teachers, they nevertheless offer us significant and insightful information for informing our teacher education practices. The issue appears not to be about what teachers know or do not know, but rather what these findings tell us about what we could or should offer teachers as mathematical experiences in teacher education initiatives.

**Learning from studies of teachers' knowledge: Rethinking mathematical experiences**

One issue that arises from these studies concerns the significance of investing efforts on teachers' development of mathematical knowledge in teacher education practices. However, as Cooney and Wiegel (2003) explain, one needs to clarify what sorts of mathematics one is talking about. Clearly, the issues are not about academic mathematics since the underlined difficulties relate to school mathematics knowledge, and academic mathematics simply appears too far removed from the mathematics teaching practices of teachers[1]. The issue therefore appears to concern school mathematics knowledge. This can appear counterintuitive, since secondary teachers are considered specialists who know their subject well and thus do not require additional education in mathematics[2]. But, the situation is more subtle than a "know/don't know" issue. What we see from these studies is that teachers do know a lot of/about mathematics. Indeed, they have had great success in their student careers and have been teaching mathematics for a number of years. Hence, even if it is reported in some of the studies that teachers have procedural knowledge or focus too heavily on algebra, this mathematical knowledge is an important part of what one needs to know in mathematics. This knowledge can be augmented, as with any knowledge, but does not need to be discarded or seen negatively as bad knowledge or in need to be erased; secondary teachers need not to "unlearn" what they know, but to continue learning about mathematics. Additionally, secondary teachers, for the most part, have a good relationship with mathematics (academically and emotionally). As Cooney and Wiegel (2003) explain, they enjoy and have strong interests for doing mathematics and investing themselves in its study. Mathematics is often seen as "part of" their lives and receives a privileged status for them: they indeed decided to pursue a career in teaching mathematics! Thus, mathematics appears as a privileged point of entry to engage secondary teachers in professional development.

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[1] There are in fact concerns currently raised about the relevance of an advanced study of academic mathematics by secondary teachers, since teachers develop professional ways of knowing and practicing mathematics strongly disconnected from what their teaching mathematics in schools (e.g., Bauersfeld, 1998; Moreira & David, 2005).

[2] An assumption that Ball, Lubiensky and Mewborn (2001) and Cooney and Wiegel (2003) question significantly.

In this study teachers were considered to possess important knowledge, and the aim was to build on teachers' mathematical knowledge and have them enhance it. The content of exploration in the sessions were to be school mathematics concepts (e.g., algebra, fractions, analytic geometry, volume, etc.), and the activities would revolve around offering mathematical tasks for teachers to explore. The specific intention of the research was to study and better understand the learning opportunities that this approach could offer these teachers.

## Methodological considerations

Ten 3-hour sessions were set during the school year, compiling 30 hours of professional development. The sessions, about diverse school mathematical topics chosen with teachers, were videotaped with the camera placed at the back of the room to capture the interactions. As the teacher educator, I designed the sessions and chose the tasks to explore (however teachers were often invited to bring tasks on their own). And, I was also an active participant in the group's discussions and explorations, as tasks only served as starting points to trigger the explorations. In short, I did not play the role of a facilitator or guide for the sessions, but invested and engaged myself in exploring the mathematical issues with the teachers.

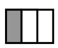
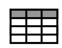
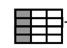
Tasks were designed on the basis of prior content analyses (didactical, conceptual and epistemological/historical analyses – Brousseau, 1998), and of content analyses derived from the research literature (e.g., on algebra, Bednarz, Lee & Kieran, 1996). In the data reported below, tasks concerned operations with fractions (+, –, ×, ÷) where the focus was placed on the meaning behind these operations (Kieren, 1976) and the relation to the whole (Hart, 1981).

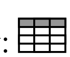
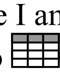
## Exploring fractions with teachers: Delving into algorithms

In order to illustrate an example of typical work/explorations done in the sessions, I report on session 7 centered on exploring operations with fractions. This topic was chosen by teachers as they were eager to investigate this topic and to know more about it, largely because they found that their students experienced difficulties with fractions and they wanted to be able to help them better. We began by exploring some addition, subtraction and multiplication of fractions with materials (egg cartons, pieces of paper and area drawings), and then shifted to division with the folding of pieces of paper and area drawings. In order to illustrate where teachers were in terms of division of fractions, here is what happened when I offered them ⅓ divided by ¼.

Carole: Me, I am not able to do this. This, I am blocked.
Jérôme: And how do you arrive at finding the answer?

Carole: Me, I know the answer. 1 and ⅓.
Jérôme: How do you arrive at it?
Carole: Technique [“multiplying by the inverse”].
Jérôme: You do ⅓ multiplied by 4 over 1?
Carole & Eric: Yes.

It appeared that teachers knew well the algorithm, but had difficulty making sense of the meaning behind this procedure. We therefore aimed to make sense of this operation. Using area drawings, the division was talked in terms of “how many ¼ enters in ⅓” – however, there could have been different ways of talking about it. So, taking a sheet of paper, a ⅓ represented the following  and a ¼ represented  or  [and ⅓ could be represented by ]. The operation became how many time 3 squares enters in 4 squares, which resulted in one and one third: 3 squares  enters 1 *and* ⅓ in 4 squares . The group appeared satisfied with the explanation, but wanted to know more about the meaning for the rest (i.e., the ⅓) and so Carole asked what would be the rest for the division ⅓ ÷ ⅕ . To begin this division, a grid was split into 5 and into 3, to which Lana, who had been very quiet since the beginning on division of fractions, objected strongly to this way of doing as we both engaged in a conversation on this.

Lana: Why do you say ⅓ ? Why do you divide by 5 first? It is the ⅓ that is divided by ⅕ . Why do you divide your sheet in fifths?
Jérôme: Because I want to know what is a ⅕ is worth.
Lana: No, but, when you read, it is ⅓ divided by ⅕ . Therefore I would start … I have difficulties; well I am not able to understand what you are all doing
Jérôme: How would you do it?
Lana: Well, I start with the first one. We did ⅓ ÷ ¼. I have drawn ⅓ like this [shows a part of her sheet divided in three parts].
Jérôme: Then…
Lana: And then I try to understand what it is.
Jérôme: That is really fine. How many quarters will enter into this [pointing ⅓]?
Lana: Yes, but what does ¼ represents?
Jérôme: Well, this is exactly it! You have to find out how much does ¼ represents. What ⅓ is worth is this [pointing the ⅓ on her sheet].
Lana: Yeah.
Jérôme: And ¼, I will try to find what it is worth. Well, if I take the same sheet and I divide it in 4 …
Lana: Ha! Ok!
Jérôme: One quarter is worth this [points to the three squares in the row:  ]
Lana & Eric: Yes !
Jérôme: Therefore I am comparing two quantities in a sense. How many times *in this* [points to ] will enter my these ones [points to ]? Well, they enter once and one third.

Lana: Ok, because, ok, ok, ok. There you have your ⅓ [looking at her sheet] and you tell yourself that you want to divide it by the ¼, but what does your ¼ represents …

Jérôme: That's it, you have to find what your ¼ is worth because it is by it that you divide.

Lana: Ok, ¼ is three squares.

Jérôme: That's it, how many times three squares enters in four squares?

Eric: Exactly!

Jérôme: One and one third.

The discussion continued, as we completed the $1/3 \div 1/5$ operation, in this way: ⅓ is 5 squares and $1/5$ is 3 squares , hence 3 squares enters 1 *and* $2/3$ in 5 squares . Following this, to step away from uniquely working with unitary fractions, we looked at the following division: ⅓ ÷ ¾. All excited, Lana shouted right away "$4/9$ ! There are 4 that fits out of the 9" with Gina being amazed by this reasoning uttering an excited "Oh!" [⅓ is 4 squares and ¾ is 9 squares , hence 9 squares enters $4/9$ in 4 squares ]. Teachers were amazed by the depth of the mathematical meanings involves in these operations and how, like Carole said, "a student that understands this will have a level of understanding of fractions that is superior." Teachers therefore started to reflect on the usual algorithm of "invert and multiply" as they knew and realized that there was much more to it. Teachers wondered why the algorithm worked and directly asked me: "why, when I divide ⅓ by ¼, I *can* invert and multiply?" I felt compelled, therefore, to offer some ways that I knew of that could help make sense of this situation. One of them concerned a glasses and litres of water context, summarized below.

> *If I have 6L of water to pour into glasses of 1L, it gives me 6 glasses of water:*
> 6L ÷ 1L glass = 6 glasses.
>
> *If now the same 6L of water are poured into ½L glasses, it gives me 12 glasses, because my glasses are two times smaller so I need twice as much:*
> 6L ÷ ½ L glass = 12 glasses.
>
> *If now I have ¼L glasses, it gives me 24 glasses, because the glasses are 2 times smaller than the previous ones of ½L or 4 times smaller than the first ones of 1L:*
> 6L ÷ ¼ L glass = 24 glasses.
>
> *If now I have ⅛L glasses, it gives me 48 glasses:*
> 6L ÷ ⅛ L glass = 48 glasses.
>
> *But if now I have glasses of ⅜L. These glasses are 3 times bigger than the ⅛L glasses, hence I need 3 times less glasses than with ⅛L glasses, which is 16 glasses:*
> 6L ÷ ⅜ L glass = 16 glasses.

What came out is that when we *divide* 6L in ½, ¼, ⅛, or ⅜, it gives the same thing *as if* we *multiply* by the inverse, namely by 2, 4, 8, or $\frac{8}{3}$. Not that "it is" multiplied by the inverse, but that it ends up giving the same answer. Therefore, 6 ÷ ⅜ is not 6 x $\frac{8}{3}$, but it gives the same answer *as if* we had multiplied by $\frac{8}{3}$. I then brought them another explanation. Since dividing by 1 is from far the simplest division that could be made, if I had to complete a complicated division, for example $\frac{5}{12} \div \frac{3}{4}$, I could use the concept of dividing by 1 or organize my operation to obtain a division by 1; by multiplying both numerators and denominators by 4/3:

$$\frac{\frac{5}{12}}{\frac{3}{4}} = \frac{\frac{5}{12} \times \frac{4}{3}}{\frac{3}{4} \times \frac{4}{3}} = \frac{\frac{5}{12} \times \frac{4}{3}}{1} = \frac{5}{12} \times \frac{4}{3} = \frac{20}{36}$$

Lana: [to Erica] I have never thought about doing that!
Jérôme: Well then, it gives me $\frac{20}{36}$.
Lana: So in a sense, you can *show* that when you divide by a fraction, it comes up at doing the inverse.
Jérôme: Yes, exactly, dividing ends up at multiplying by the inverse.

Gina explained to the group that these explanations, in addition to give meaning to the algorithm of "invert and multiply," made her realize that *it was* indeed a trick, something she had never saw in these terms before, and had mostly seen it as a blunt fact. Teachers continued commenting on the reasoning behind the calculations we had completed, as the session was coming to an end.

**Reflecting on teachers' knowledge development**

The above excerpts are illustrative of comprehensions teachers developed during the year about school mathematics (here, for operations on fractions), concepts they knew and used frequently, as they worked and re-worked in deeper details these concepts and thus expanded their understanding of them. Davis and Simmt (2006) call this process *recursive elaboration*, as one refines and re-elaborates elements of personal knowledge. This is what happened here, where teachers' knowledge was both deepened and refined. In addition, these concepts also changed for the teachers. When one compares the meaning they previously gave to division of fractions and what they appeared to have developed, one sees an important change that has happened. Even more, Gina's realisation that the algorithm or trick to divide fraction was in fact a trick, something she had never thought of before, made her change significantly her view of the concept. Lana's comment that she never had thought about ways of showing how why the algorithm works and offering a sort of proof is also speaking of this significance. In fact, towards the end of the session, both Lana and Eric came to tell me that they had never thought about explaining and doing division of fractions in these ways and would certainly use these explanations in their teaching (Eric even added: "In fact, tomorrow morning!"). What appears significant here is that these teachers not only gained some mathematical understandings about the division of fractions algorithm, but also envisioned possibilities to communicate this understanding to their own students and re-invest it in their teaching. From this we can see an explicit link between their knowledge and teaching of mathematics. New mathematical understandings appeared to open a space for new teaching possibilities and ways of working with concepts (different than only offering the algorithms).

## Concluding remarks and implications

These mathematical amazements happened continuously throughout the year, as teachers explained that the explorations gave them access to ways of understanding mathematics that they had never experienced before in their education and were unfamiliar to them. As Carole kept repeating to me and other in the sessions and when I visited her informally in her school "you completely changed my teaching!" That this be true is not up for the test, but what strikes me more importantly is the fact that the only thing we did was to engage in school mathematics concepts, nothing more! It makes this approach focused on school mathematics very promising for teacher education practices, as there is the opportunity to offer mathematical experiences that some teachers have simply never experienced before.

To conclude, one thing this study makes clear, in addition to the learning of mathematics and to pedagogical reflections, is that secondary teachers *can* learn a lot of mathematics. What this indicates is that even if research studies point to difficulties or misunderstandings secondary teachers may have, they have the capacity to learn more mathematics and continue evolving in mathematics. The fact that these explorations opened whole new mathematical horizons for these teachers is also of significance. As Even (1993) and Hitt-Espinoza (1998) explain, teachers often simply never had the opportunity to engage with these sorts of mathematics that go beyond a single usage of procedures. This is very encouraging and significant for our practices of mathematics teacher education, where we have the opportunity to push teachers' mathematical knowledge forward. In a time where concerns are raised about the disconnection between teacher education and teachers' teaching practices in school, an approach focused on teachers' knowledge of school mathematics appears relevant since it is directly related to their teaching practices as they develop professional ways of knowing and practicing mathematics.

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