Mathematical practices and new potential instructional trajectories in a dynamic environment

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In this paper, we argue that the systematic use of digital tools offers teachers and students multiple opportunities to develop mathematical thinking. In this context, we sketch elements of a framework to characterize and analyze the emerging instructional mathematical practices and students' problem solving approaches that appear in mathematical scenarios that foster the use of the tools. Thus, the expressive nature associated with the use of the instruments to represent and deal with mathematical objects or problems not only favors new routes to visualize and explore mathematical relations; but also ways to provide or support those relations. In this context, the outlined framework becomes useful for researchers to investigate new research questions and to orient teachers in their instructional practices.

Introduction: Mathematical activity in a digital world

Every national educational system, in one way or another, is designing programs or taking actions to face the future with education as the sustaining foundations of what is called *the century of information and knowledge*. Every country is facing new challenges in education, which entails the need to reformulate what is taught, as well as how and why it is taught. Eventually, we are forced to ask: What is the *new* role of mathematics in contemporary societies? To be precise, one would have to take into account the degree of development of the society under study, but the pervasive presence of digital technologies continues to remind us that these technologies *will rock our worlds*. Thus, we foresee important transformations in the field of curricular design and development, as well as in the application of new learning tools.

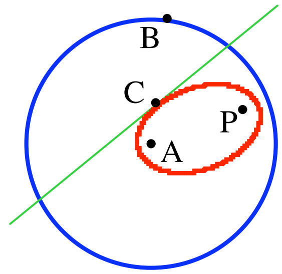
Coming to contemporary classrooms —the kernel of any educational system—, we cannot forget that a pre-existing school culture has left its marks on the values and conceptions that implicitly and/or explicitly have a significant influence in the educational system. These values were established, through history, in environments poor in technology, and they have only slowly come to terms with the evolution of mathematical practice linked to technological evolution. Thus, the school culture requires the gradual re-orientation of its practices and of its cognitive and epistemological assumptions to gain access to powerful ideas of mathematics and to new habits of mind including exploring, modeling, handling of information, and the ability to systematize.

It is possible to gradually cultivate powerful mathematical ideas, mediated by a dynamic environment that generate different levels of mathematical thinking. A Dynamic geometry Environment (DGE), like Cabri, provides an environment that estimulates mathematical expressivity. Students can explore their tasks taking profit from the executability embodied in the environment and the malleability of the digital representations. The latter is not arbitrary: it reflects the structural possible variations of the figures. This means that explorations are subjected to structural constraints and consequently, students *will be guided* by the environment in their development of geometrical (mathematical) perception. Now, from being guided by the environment to guide the environment, is the result of *co-acting* with the environment. This is a fundamental feature of artifact mediation. To illustrate further, let us imagine a cellist. The cello is like an extension of herself in the sense that while playing, the cello is t*ransparent*. The artist can feel the music *through* the

cello; the cello is not an appendix to the artist, it partially *defines* the artist. It must be said that at this level, the cello has been developed as a model of musical interpretation guided by the artist. This dialectic put into motion by the *co-action* of the agent and the environment, is similar in the case of a DGE. Mathematical expressivity reflects the control that eventually the student gains over the environment. The cognition of the artist (student) is transformed: her art is not the result of doing something better, something that she could do without the cello per se, but something intrinsically linked to the new activity that emerges from the new dialectical interactions cello-cellist. Instead of been guided by the cello, the artist is *guiding the cello*. This is a multilevel process.
Our work involved the design of tasks focusing on students' development of reasoning skills that deal with the construction and verification of hypothesis or conjectures. The expectations have to do, in particular, with providing students with the opportunity of (a) experiencing the construction and development of mathematical knowledge; (b) developing their own mathematical expressivity through the mediation of dynamic environments.
The dynamic environments enable the exploration, conjecturing, systematization and justification of mathematical relations. Dynamic representations of mathematical knowledge are instrumental to facilitate the students' construction of cognitive tools (knowledge artifacts) to identify and explore mathematical relationships. Finally, the presence of digital technologies requires the understanding of problems of transparency and opacity: cognitive and epistemic, that the technology in question needs to address.

The knowledge embedded in a dynamic environment

As has already been said, dynamic environments enable students to explore with *executable representations* of mathematical objects. These representations are structural and reflect invariants intrinsic with the *dynamic versions of mathematical objects*. Let us consider an activity from the classroom where students constructed dynamic representations of conic sections and explored different loci resulting from the dragging of particular elements of the representations of the conics. It is well known that conic sections constitute an important example in geometry. Here and in the following sections, we will present a set of problems leading to the construction of the conic sections, and *the use of locus and trace* as semiotic mediators whilst solving the problems presented. The semiotic mediation of the *locus* is fundamental. Dragging objects, analysing behaviours, formulating conjectures, presenting arguments and communicating results seem to be important activities that students can practice systematically with the mediation of dynamic geometry environments, in order to detect and explore mathematical properties. Mathematical objects are *evolving* objects that students explore while increasingly using more refined and robust resources and strategies. All this meets deep *cognitive and epistemic problems* that cannot be avoided or dismissed in the classroom. In traditional educational systems, these problems rarely surface during educational practices but now, the mediation role of the dynamic environment, makes them tangible and visible.
Students have access to conjecturing and generalizing by clicking and dragging *hotspots* on an object which dynamically re-draws and updates information on the screen as the user drags the mouse. In doing so, the user can efficiently test large iterations of the mathematical construction. The figure below, attempts to illustrate this dynamism through a snapshot of such a physical action. The environment updates the figure whilst the user drags point B (the locus of point C is an ellipse, in this case).



What is the locus of point C (intersection of the perpendicular bisector of PB and line AB when point B is moved along the circle?

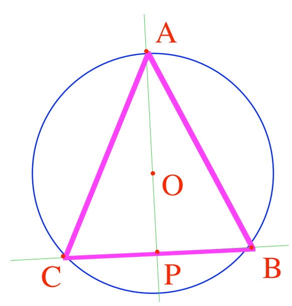
The media can keep a trace of such constructions and actions, but more so, co-actions between the user and the environment. As mentioned already, such embodied actions of pointing, clicking, grabbing and dragging parts of the geometric construction allows a semiotic mediation between the object and the user who is trying to make sense, or induce some particular attribute of the diagram or prove some theorem. The dynamism of the user's actions is crystallized within the geometric diagram.

The artificial realities of the diagram obey the rules of geometry that are preserved in the elements of the diagram, just as world objects obey the rules of physics in nature (Laborde, 2004). But when an element of a diagram is dragged, the re-constructions are re-enacted by the environment, not the user.

The Voice of the Students

Some interesting cognitive behaviours are shown in an interview with Estela, a 16-year-old high school student. She is a bright student with a good working knowledge in mathematics. The instructor poses the question concerning the largest area of the triangle in these terms:

An isosceles triangle is inscribed in a circle; one of the triangle's vertices A, coincides with an endpoint of a diameter and the remaining two vertices B, C, are the endpoints of a chord BC that is perpendicular to this diameter.



Searching for the largest area.

The instructor makes clear that the point P can be moved along the diameter. When the point P moves along this diameter, the chord BC also is moved and remains perpendicular to the diameter.

Instructor: When I move point P what happens to the triangle? Does its area change? Does it remain the same?

Estela: The shape changes…it seems that the triangle remains as isosceles…but its area is the same.

Instructor: Can you explain?

Estela: Because here (pointing to the sides AB and AC) we are making the triangle smaller and here (indicating BC) we are making it larger…

The instructor continues, asking if there is a position of point P wherein the area changes, where the area is smaller or larger, and the student's response is:

Estela: No, the area remains the same…(Then she draws, on the slate, a triangle with BC very short)…for instance in this triangle, BC is smaller but AB and AC are much larger…

The instructor asks Estela to construct the figure on the screen. Afterwards, she begins displacing point P (upward) along the diameter without reaching the center O of the circle.

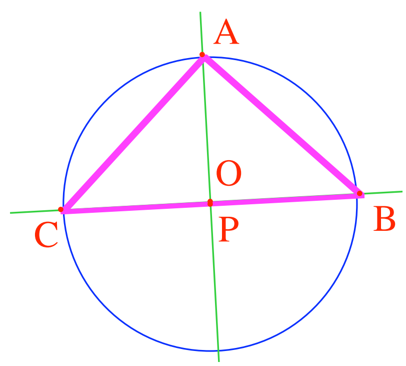
Instructor: Does the area change?

Estela: No…it does not.

Instructor: Why?

Before answering, Estela displaces P, very slowly, without reaching O.

Estela: If the side BC grows, the sides AC and AB are decreasing…



Looking for conditions of area changes.

Instructor: The area of the triangle…does it ever change?

Estela: No, because…

Then, Estela moves P towards O and, *for the first time during the interview*, P goes beyond O until almost touching A, and then drags P back beyond O. *At that very moment she discovers something new*, and says:
" When I drag P beyond O, AB and AC gets shorter and BC also gets shorter…" (Estela seems concerned and she drags P again. It is clear that something is disturbing her, attracting her attention. *Her perception is changing*...)

Instructor: What is going on?

Estela: I was looking at the area…it…it decreases*, the area decreases*!

Instructor: When does it decrease?

Looking at the chalkboard she says:

Estela: If I drag P beyond O, the area is smaller.

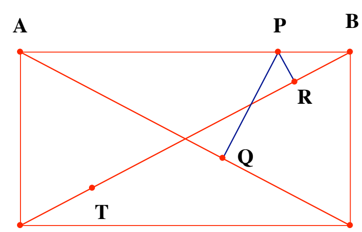
After the interview, Estela remarked: "I discovered that the area would decrease because, if the base BC and the sides AB and AC are all smaller (than previously), then the area has to be smaller. At the same time, if I drag P from A, without reaching O, the BC gets larger, AB and AC also gets larger, so the area grows." Estela has seen the simultaneous variation of three magnitudes when she displaces P along the diameter. Yet what she observes on the screen is not the variation of the area itself (which cannot be *seen*) but an *index* (Deacon, 1997, p. 71) of this variation, a sign that triggers an inference: *the area becomes smaller if the sides become shorter*.

The reference field of a symbol has a potential infinitude of levels of reference. Some are iconic, others are indexical. Sometimes Estela is looking *at* the (static) figure, sometimes, later, she is looking *through* (the movement) the *executable* representation, a figure that finally *reveals* what she has been hiding.

We said that Estela perceived on the screen an *index of the variation of the area* that triggers an inference. This inference is her first level of understanding of the variation of area. Here is where her perception goes beyond a mere act of "encounter the drawing with the eye." It becomes an *act of visualization*, that is, a sensorial phenomenon controlled by a semiotic interpretation.

Exploration and infiltration of proofs

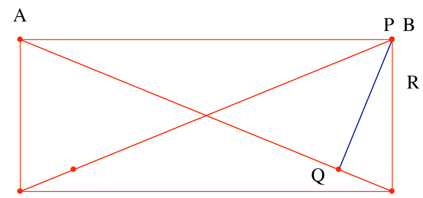
Imagine a traditional classroom confronted with this problem:



Prove that Prove that the sum of the length PR and PQ (perpendicular projections to the diagonals of a rectangle) is a constant.

This is the kind of problem that left students perplexed. A constant? What constant? And there they go. However, when the problem is posed in a dynamic environment (a classroom can become a dynamic environment), the *expressive capacity* with which students are empowered thanks to the executability of the dynamic system of representation, eventually takes them to the realization that dragging the point P along AB and close to A, reflects some symmetry hidden in the problem. *They can stay close to B*. This problem was solved, along these lines, by students in an undergraduate Mathematics Inquiry course for seniors at an American University. With a paper and pencil environment, students were not able to formulate a conjecture. The dynamic environment transformed the mathematical identity of the students who were able to share ideas and criticisms openly.

Now, staying close to B, suggest, eventually, taking P to the position B. This has the effect of revealing a conjecture: the constant is BQ (see figure below).



The conjecture is revealed

The expressive capacity and *the plasticity* of a dynamic environment, induce new strategies to explore and conjecture.

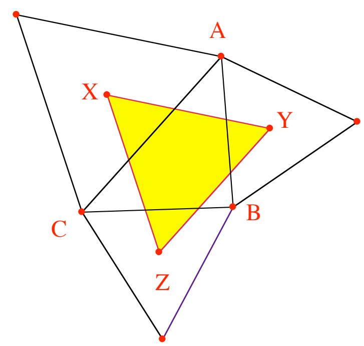
Formalization and rigor are relative to the media in which they take place. If we use digital semiotic representations of mathematical objects, what are the valid rules *to prove—*that we are allowed to use in the new digital environment? This methodological problem involving semiotic representations of mathematical objects has been considered in mathematics, in the past. We can find proofs in the works of the greatest mathematicians, Euler for instance, that could not be published today. This happens because *paper-and-pencil mathematics*—mathematics as we know it—has been continuously refining its own standards of proof.
How is all this to affect the mathematics (of mathematics education) in the future? The inherited corpus of shared mathematical knowledge produced in interaction with pre-digital technologies is large and stable (Kaput, Hegedus and Lesh, pp. 174, 2007). So we need to create *transition strategies* to transform basic contents of this stable corpus of mathematical knowledge into the new digital semiotic supports. Here we are still working at the *border* between the paper and pencil (classical) epistemology and the new digital (applied) one.
As Rotman (2000) has forcefully asked:

> Is it unnatural or deviant to suggest that immersion in a virtually realized mathematical structure...be the basis for mathematical proof? Would not such proofs, by using virtual experience as the basis for persuasion, add to, and go far beyond, the presently accepted practice of manipulating ideograms and diagrams in relation to an always invisible and impalpable structure? (pp. 68-69).

The problem with the determination of mathematical objects comes from the fact that we cannot exhaust the set of its possible semiotic representations. The mathematical object is therefore, always, under construction. It is an *unfinished object*. And, at the same time, it has unexpected properties that traditionally, have been put explicitly, as theorems. It is through this tension that the mathematical object is constituted. It is not surprising then, that a thinker of the stature of Rene Thom (1972) had expressed in his plenary lecture at ICMI 2:

The real problem that confronts mathematics teaching is not that of rigor but the problem of the development of meaning, of the existence of mathematical objects.

Let us introduce an example to illustrate how we can *infiltrate* the classical methodology of proof. Let us consider the Napoleon Theorem: *Given any triangle, we construct the corresponding equilateral triangles on the sides of the original triangle. The triangle obtained by connecting the centers of the equilateral triangles previously constructed, is always an equilateral triangle*. Such is the content of this classic result. It is very easy to persuade a student of the validity by dragging any vertex (A, B or C) of the original triangle:



The shaded triangle is Napoleon Triangle. Dragging any vertex A, B or C indicates that the napoleon triangle is equilateral; *the activity crystallized in this dragging is indexical*. But there is more knowledge already embodied, infiltrated, in this semiotic representation. To make a long story short, let us describe how to proceed to obtain a valid proof **within** the dynamic environment. Given X and Y, we can construct the third vertex of the equilateral triangle whose side is XY. When we place the pointing device close to point Z, we will receive the question: What point? Because the recently constructed point is occupying the same position as Z. Can we separate these two points *in the environment*? No, we cannot because both are dependent on previous constructions. Being non-separable means they are the same point. This proves that the Napoleon triangle, XYZ is equilateral. This is a new culture of proof and it is up to our community to decide the place this kind of infiltration will have in the future. We might be touching a very delicate epistemological locus. It is timely to remind ourselves of Rotman's (2000) reflections on this issue:

> ...Such a transformation of mathematical practice would have a revolutionary impact on how we conceptualize mathematics, on what we imagine a mathematical object to be, on what we consider ourselves to be doing when we carry out mathematical investigations, and persuade ourselves that certain assertions, certain... a "theorem" for example would undergo a sea change (p. 68-69).

References

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