Experimental mathematics and the teaching and learning of proof[1]

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Abstract

Aim of this paper is to discuss the role of experiments in mathematics for the teaching and learning of proof. Experimental approaches are more and more popular in the literature on didactics of mathematics, under the pressure of the explorations made possible by Information and Communication Technologies (e. g. Dynamic Geometry Environments, Computer Algebra Systems), which allow students to collect in a very short time many experiments and to look for generalizations. Teachers, on the contrary, show doubtful about the advantages and, very often, complain that students have lost the motivation to construct proofs. To enter the debate, I shall summarize some research findings from basic research studies and from teaching experiments. The examples comes from teaching experiments at all school levels on space and geometry by means of classical resources (and not ICT) although some of the findings might be expanded to other subject areas. They allow to frame the topic within the international literature on conjecture production and proof construction: they support the advantages of experimental approaches to the teaching and learning of proof and, at the same time, point at some critical points to be controlled in order to design appropriate teaching interventions.

Introduction.

A growing interest is shown, at the international level, for the development of approaches to mathematics where the active participation of students is encouraged within a laboratory setting, with hands-on activities. The emphasis on experiments, manipulation and perception, measurement and examples is shared by the approaches developed within ICT environments (both DGE and CAS) and within classical technologies (straightedge, compass and ancient instruments). This experimental approach, where exploration plays a major role, seems appealing for students, who quite often find the evidence offered by a particular experiment much more convincing than a rigorous proof (Jahnke, 2007) and are bored by the request to produce also mathematical arguments. Hence, the appeal of experimental approach might be suspected of obstructing the development of mathematical styles of reasoning: some believe that hands-on activities are useful in either science centres or mathematical festivals, where popularization of mathematics is in the foreground, whilst are not useful and may be even risky in the mathematics classrooms, where the construction of mathematical meanings is at stake. In other words, many mathematics teachers are afraid that the need of mathematical proofs and of deductive arguments is put in a difficult position if experiments are given too much space in the mathematics classroom, at least in secondary schools.

One might object, for instance, the existence of a specific academic journal, named "Experimental Mathematics" (http://www.expmath.org/), devoted to experimental aspects of mathematics research, that publishes formal results inspired by experimentation, conjectures suggested by experiments, descriptions of algorithms and software for mathematical exploration, surveys of areas of mathematics from the experimental point of view. Also the "confessions" of professional mathematicians who describe how new ideas in mathematics are developed (e. g. Thurston, 1994) might put experiments back in right perspective. Yet, it is doubtful that these objections may convice teachers, without examples of good practices in the mathematics classroom where the experimental approach is dialectically intertwinned with the construction of mathematical proofs. In the following, after a short review of literature, I shall present some effective experiments at all school levels where experiments and exploration have been combined with conjecture production and proof construction.

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Some studies concerning proving in the mathematics classroom.

The literature on proof and proving is large and encompass different aspects. In the recent book on "Theorems in School" edited by P. Boero (2007), the following aspects are highlighted: the historical and epistemological dimension; curricular choices, historical traditions and learning of proof (including two national case studies); the cognitive dimension of the relationships between argumentation and proof; the didactical dimension including both teacher education and classroom practices.

In the chapter authored by Bartolini Bussi et al. (2007), a *mathematical theorem* – for didactical purposes - is conceived as a system of statement, proof and theory. All these three components are important: the theory as a system of shared principles (sometimes called postulates or axioms and definitions); the statement as the result of a conjecturing process, where exploration through experimental activity is in the foreground; the proof as a sophisticated argumentation that is, on the one hand, connected with the conjecturing process, and, on the other hand, consistent with the reasoning styles of mathematicians (e. g. deduction from the accepted principles). This approach is consistent with Jahnke (2007), who speaks about 'local theories', i. e. small networks of theorems based on empirical evidence and claims: "There is no easy definition of the very term ''proof'' since this concept is dependent of the concept of a theory. If one speaks about proof one has to speak about theories, and most teachers are reluctant to speak with seventh graders about what a theory is". And Arzarello (2007) adds: "A statement B can be a theorem only relative to some theory; it is senseless to say that it is a theorem in itself: even a proposition like "2+2=4" is a theorem in a theory A (e. g. some fragments of arithmetic)".

In the above sense, it is possible to speak about theorems also within primary school, provided that the theories are "germ theories", drawing on empirical evidence, with the expansive potential to capture more and more principles. Germ theories, with principles constructed on empirical evidence, are crucial up to 8^{th} grade; later, accordingly to curriculum, the reference to more and more structured mathematical theories is possible. So, for instance, in the teaching experiments below, the reference theory from grade 11th on is expected to be elementary geometry (either 2D or 3D) with some additional parts concerning either isometries or conic sections.

The links between argumentation and proof from a cognitive perspective have been carefully analysed by Pedemonte (2007) who devoted her doctoral thesis to the development of the idea of *cognitive unity,* meant as a kind of continuity between the production of a conjecture and the construction of the proof. Experimental research shows that proof is more 'accessible' to students if an argumentation activity is developed for the production of a conjecture: in fact this argumentation can be used by the student in the construction of proof by organising in a logical chain some of the previously produced arguments. These studies may have important consequences on the teaching and learning of proof: to explain why rote learning of ready made proofs is not successful for most students; to select suitable problems, which might foster conjecture production before proof construction; to understand why in some cases proving remains difficult in spite of the previous conjecturing process.

In the following sections I shall quote and analyze very quickly some experiments where conjecturing and proving were promoted, at different school levels and with different organization.

Examples from long term teaching experiments.

In the following table, some paradigmatic examples are quoted from long term teaching experiments developed as coordinated studies by different research teams. All the tasks concern a conjecture production before proving construction. They appear, however, different from each other.

Grade	GERM THEORIES (REF)	CONJECTURES - PROBLEMS *THE TASK - TO BE SOLVED IN WRITING*	SETTING MATERIAL
1. Gr.2 - 8	The invariance of alignment in perspective drawing (Bartolini Bussi, 1996)	The centre of a table drawn in central perspective. *Draw the small ball in the centre of the table. You can use instruments. Explain your reasoning.*	Individual task (Fig. 1)
2. Gr.2 - 8	Motions of geared wheels (Bartolini Bussi et al., 1999)	The motion of trains of toothed wheels. *What about three wheels geared with each other?*	Individual task No material
3. Gr.4 - 8	The equality of the distance of the centres of two tangent circles to the sum of radii (Bartolini Bussi et al., 2007)	The drawing of a circle tangent to two given circles. *Draw a circle with a radius of 4 cm tangent to the given circles (radii 3 and 2). Explain carefully the method. Explain carefully why it works.*	Individual task (Fig. 2)
4. Gr.6 - 8	Mathematical model of sunshadows. Basic properties of lines, planes, parallelism and perpendicularity (3D geometry). (Boero et al., 2007)	The parallelism of sunshadows of sticks. *In recent years we observed that the shadowsof two vertical sticks on the horizontal ground are always parallel. What can be said of the parallelism of shadows in the case of a vertical stick and of an oblique stick? Can shadows be parallel? At times? When? Always? Never? Formulate your conjecture as a general statement.*	Small group work. Pens, pencils, notebooks, rulers, to reify lines and planes
5. Gr.11	Elementary geometry (3D geometry). Definitions and properties of isometries. (Bartolini Bussi & Pergola, 1996)	The isometry (rotation) produced, as a correspondence, by a pantograph. After a guided exploration of the pantograph. *If P and P' are two writing points, draw two corresponding figures. Which are the common properties of the two figure? Can they be superimposed? Does it exist a simple motion which superimposes them? Describe it.*	Small group work. A pantograph with graphite leads in P and P' (Fig. 3).
6. Gr.12	Elementary geometry (3D geometry). Metric definition of conics. Equations of conics (Bartolini Bussi, 2005)	The conic obtained by cutting a cone in a suitable way.The task is given orally by the teacher. *You have to obtain an important property of parabola [...]. As you see,* [the parabola] *is in a 3D space, on the surface of the cone [...]. you have to discover the relationship between the green line segment* [AE in the Fig. 4] *and this line segment* [EB in the Fig. 4].	Small group work. A 3D model of a cone with a normal cutting plane (Fig. 4).

SETTINGS

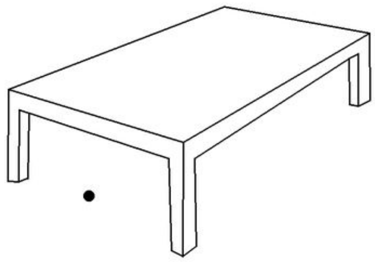

Figure 1
The small ball and the table

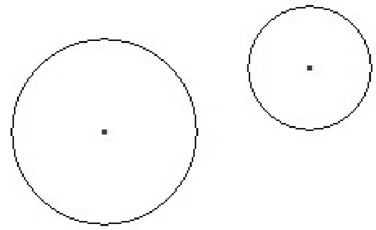

Figure 2
The two circles and the tangent circle

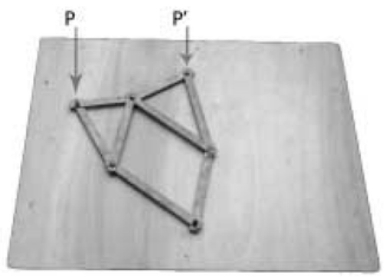

Figure 3
The pantograph

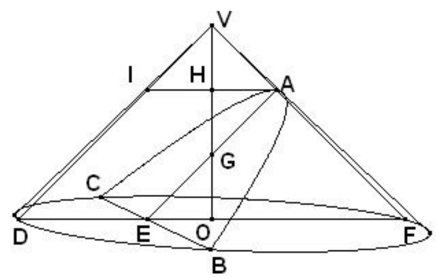

Figure 4
The parabola

Three tasks (tasks 1,2,3) concern individual activity, to be solved in paper and pencil setting; three tasks (tasks 4,5,6) concern small group activity, to be solved in writing after the exploration of a material object. The exploration is free in the case of sunshadows (task 4), whilst it is guided by worksheets or by the teacher himself in the two cases from secondary school (tasks 5 and 6).

The tasks 1 and 3 are construction problems: they require to produce a drawing and to justify the validity of the used method. The expressions "Explain ....." mean, in a language accessible for young learners, to justify the drawing process with reference to a shared (germ) theory. The task 2, on the contrary, seems to be given in a discursive way. Yet the explanation requirement with reference to a shared (germ) theory is implicit, as a part of the tacit rules shared within the classroom involved in these experiments.

In the last three tasks proof is not explicitly required. Actually the focus is on the production of the conjecture. This is an intentional choice, because the problems are quite demanding. The tasks 4 and 6 concerns 3D geometry, that is usually not well mastered by secondary school students. The task 5 is difficult: the conjecture concerns a rotation around the lower point (O) in the Fig. 3. Actually to recognize it, it is necessary to "see" two line segments (OP and OP') that do not exist, to realize that they are always equal and, more generally, to be able to "see" invariants during the motion. The teachers, for the tasks 4, 5 and 6 had designed, according to the shared theoretical framework, an intermediate step where to collect and discuss the conjectures, before entering the proving process.

In the task 4, students are explicitly requested to produce a general statement. This expression was used in those classrooms to foster the production of statements with universal quantifiers (all, always, and so on) and hopefully in conditional form (if ... then) to pave the way towards the construction of a proof with specified hypothesis and thesis.

Concrete exploration is encouraged in the tasks 4,5,6 and not clearly mentioned in the tasks 1,2,3. This might surprise people who think that the mastery of the three modes of intellectual development (*enactive, iconic and symbolic*, Bruner, 1966) is related to the learner's age. On the contrary, according to Boero et al. (2007), at all ages, the dynamic exploration of a suitable problem situation has a crucial role both at the stage of conjecture production and during the proof construction. In particular, as to the conjecture production "the conditionality of the statement can be the product of a dynamic exploration of the problem situation during which the identification of a special regularity leads to a temporal section of the exploration process, which will be subsequently detached from it and then "crystal" from a logic point of view ('if .... then')"; and as to the proof construction, "for a statement expressing a sufficient condition ('if ... then'), proof can be the product of the dynamic exploration of the particular situation identified by the hypothesis" (Boero et al, 2007, p. 249 ff.). This phenomenon has been observed by Boero et al. (2007) for the task 4 about sunshadows, by Bartolini Bussi & Pergola for the task 5 about the pantograph (Bartolini Bussi & Pergola, 1996) and in other ongoing experiments on either transformation or curve drawing devices. As concrete manipulation of materials is not spontaneous and guaranteed with elder students, who had already spent years to learn (or better to be taught) that mathematics is just a mental activity, the teacher has to foster it in a very coercive way: concrete exploration in demanding tasks is quite often the only effective way to promote dynamic exploration. Younger pupils, on the contrary, are accustomed to explore and to evoke exploration when no concrete object was available.

The processes.

The six situations above, although in different modes, have been designed to foster cognitive unity between the conjecturing and the proving phases. I shall not try to summarize here the observed processes concerning them all: they are complex, long standing, different (also for students' age) and all available in the international literature. Rather I shall illustrate another simple case of conjecture production and proof construction at secondary school level (from grade 10 on, according on the curriculum), concerning a curve drawing device (Maschietto & Bartolini Bussi, submitted). I shall narrate the stories of dynamic exploration that show up when secondary school students are given this curve drawing device to foster reasoning, conjecturing and proving (another example is discussed by Bartolini Bussi, in press).

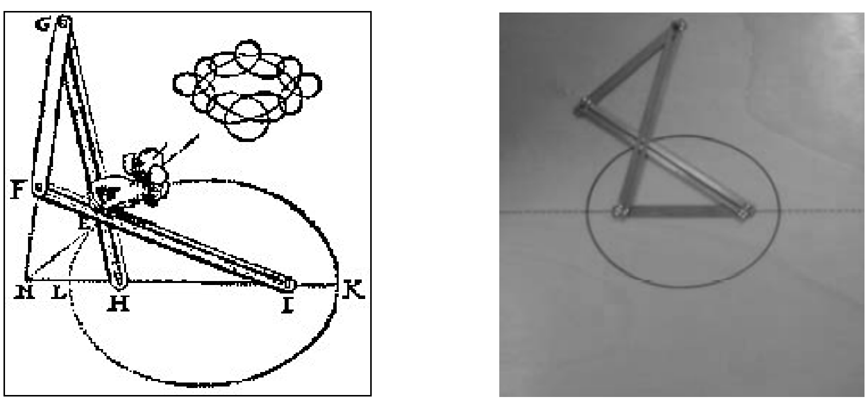

Fig. 5a and 5b

I shall collect some evidences from the field notes of the exploration sessions in both school classrooms and the Laboratory of mathematical Machines[i], to highlight the patterns that emerge. The two parts of the fig. 5 show (on the left) a drawing from the XVII century treatise by van Schooten (1657, p. 339) and (on the right) a photo of the brass copy reconstructed on a wood platform (40 cm x 40 cm) by the team of the Laboratory of Mathematical Machines at the Department of Mathematics of Modena, to be used with secondary and university students. The students are supposed to know some early properties of conics, e.g. the string and pencil drawing of an ellipse (together with the ellipse metric definition). There are several ways to explore the artefact (in order to produce a conjecture and to construct a proof of the conjecture) that span from strongly to weakly guided ones. In general, strongly guided exploration is suitable to a laboratory short term sessions (at most 2 hours, including the introduction and the conclusion of the hands on activity, Maschietto & Martignone, in press), whilst weakly guided exploration is suitable to classroom activity, when the teacher plans to spend more time on the same topic. Actually with a weak guide, the time may expand, not matching the time constraints of a short visit to the Laboratory.

A) *Strongly guided exploration*. Students are given a worksheet where a layout of the artefact is drawn with coding letters (examples: http://www.mmlab.unimore.it/on-line/Home/VisitealLaboratorio/Materiale/articolo10005163.html) and are suggested to identify the fixed points, the trajectories of the moving ponts (e.g. G and F), the length of the bars, and so on. After this exploration, they are asked to conjecture the name (if any) of the trajectory of the point E (intersection of GH and FI in the fig. 5a) tracing it with a graphite lead on the wooden platform. The drawing is soon recognized as an arch of an ellipse and the conjecture is produced. Then the process of proof construction is to be started. We shall comment it later.

B) *Weakly guided exploration.* Students are given the artefact and the information that it may draw curves; they are given the burden to produce conjectures and to prove them. A graphite lead to trace the trajectory of points is available with no special emphasis on this experiment: they can decide to use it or not. The artefact is without coding letters (Fig. 5b) and actually the need of coding may be one of the outcomes of the exploration to understand each other (Bartolini Bussi & Pergola, 1996). When the students explore for some minutes the motion without drawing the arch, they may recognize a well known (although hidden) figure. HIGF (fig. 5a) is an isosceles trapezium with diagonals (HG and FI) and sides (FG and HI) given by brass bars, whilst the bases FH and GI have a variable length and are not reified by bars. The figure is not trivial to be noticed, as the two bases are not visible. Usually the students rotate G around H and observe the figure. Sometimes they seem fascinated by this rotation and stay silent for minutes. Sometimes they close their eyes to follow, maybe, a mental experiment. They try to look at the artefact from different perspectives, also standing and moving around the table. They assume strange postures, twist their necks to follow the motion, point at the bars and follow the motion with the finger in the air, move the bars forward and backward to look for invariants and test them stopping the continuos process. In the small group work, sometime a conflict arises, when the speed of the motion controlled by the actor does not match the exploration planned by the observer. At one point they "see" the trapezium and notice that EG = EI and FE = FH. When a student has "seen" the trapezium, this figure is immediately shared with others. When the trajectory of E is eventually drawn they have at disposal what they need to link the conjecture with the metric property of ellipse.

I have described two 'antipodal' exploration processes with a lot of mixed cases in between. The weakly guided one is enjoyed by experts. The strongly guided one suits novices' needs to avoid frustration: it aims at encouraging to handle the artefact and at scaffolding the process. In both cases the demanding part is not the conjecture production, especially when drawing by the graphite lead is encouraged. Actually, as soon as the user draws the curve, the conjecture springs up, because only a limited set of curves is known by students: it is neither a circle nor a parabola nor an hyperbola, hence it must be an ellipse. The demanding task in this case concerns proof construction. This situation is different from the one of the tasks 4 and 5 above, where also conjecturing is really demanding.

In the strongly guided exploration, the worksheet suggests some ways to explore the properties of the artefact. Yet, in order to notice the properties, measuring by rulers is suggested. Measuring requires to stop the motion and to transform the experience of continuous motion into the observation of a finite set of frames. The focus risks to be on measuring parts of a still figure.

In the weakly guided exploration, the focus shifts on the observation of dynamically changing shapes and their invariants. The students have to move and observe. Their process seems time wasting and not effective and has to be monitored by a walking teacher who moves from one group to another showing how to explore the artefact, with changing speeds and, maybe, no word. The initial 'weak' guide seems to require a stronger teacher's control. The students do not need (and usually do not wish) to measure bars by a ruler. As soon as they notice some invariants, they use their hands: they pretend to pick up the line segment EG between forefinger and thumb and to rotate it until it matches EI. They repeat the action on the pair FE and FH. Silent gestures seem to be enough to convince them. Maybe words and deductive chains are missing. Writing and justifying (by symmetry, for instance) the equality:

$$HE + EI = HE + EG = HG$$

that represents the metric property of ellipse with foci H and I is the boring counterpart of a relationships discovered by making "infinitely many" experiments, during the continuous motion of G around H.

In both cases of exploration, if the drawing is produced too early, the attention is focused on the final result of drawing rather than on the dynamical process of drawing. I shall consider this later.

There is a difference between the strongly guided exploration, that foster the production of statements concerning pointwise construction of the trajectory and the weakly guided exploration, that foster the production of statements concerning the global construction of the trajectory by a continuous motion. This difference is epistemological and mirrors the ancient pointwise construction of curves and the modern (as from the 17th century) construction of curves by a continuous motion of a machine (Lebesgue, 1955). In the pointwise construction, there is a gap between the statements concerning a particular point E obtained when the artefact is in a given position and the generalization to a whichever point of the trajectory. This gap might obstruct the proof construction, requiring additional arguments: for instance the students must be aware that the statements produced in one frame concerns a "generic" point rather than a particular one and control the situation.

The situation is different, yet recalls the one analysed by Pedemonte (2007) and concerning the construction of proofs by mathematical induction. She analysed the sum of the interior angles of an n-sided convex polygon, but the reasoning might be applied to many cases of induction. The well known formula: (n - 2) times 180°, may be conjectured in at least two ways, that draws on experimental activity and that are called: result pattern generalization (the cases of n-sided convex polygons are analysed separately, adding the measures of the interior angles, for n=3, n=4, n=5 and so on); process pattern generalization (from an (n-1)-sided convex polygon, for n=3, 4, 5 and so on, a new n-sided convex polygon is obtained by the juxtaposition of a triangle, whose sum is 180°).

The result pattern generalization does not help much to construct the proof by mathematical induction, because the constructive argumentations used have no counterpart in the proof. On the contrary the process pattern generalisation paves the way towards the proof, showing how it is possible to shift from n-1 to n. Pedemonte (2007) says that in the second case there is a structural continuity between the conjecture production (by argumentation) and proof construction (by induction). Students may succeed in proving the conjecture also after a result pattern generalization, but they must reconstruct a suitable argumentation that links the conjecture to the proving process (called by Pedemonte, 2007, a structurant argumentation).

The shift to the analytic frame suggested in the Laboratory worksheets is an intentional break of the structural continuity, because the analytic frame is supposed to be the familiar context where conics are studied in secondary schools.

Discussion

Some conclusions may be drawn from the quoted examples and research outcomes. They concern exploration by means of classical resources (ruler, compass, drawing instruments and so on), but there is a parallel development of research studies with consistent frame and results in the subject area of Information and Communication Technologies (e. g. mariotti, 2001).

First, there are good reasons to believe that conjecturing through exploration before proving might be very useful. Yet, when conjecture production is too fast, it might offer no element to be used in the proving process. Hence it is useful to look for strategies that slow down the conjecture production and encourage effective exploration of the problem. The time spent in conjecture production is not wasted and may be recovered in the proof construction.

Second, it is not possible to give general rules about which exploration is effective in the conjecture production. In the last example, I have contrasted strongly guided and weakly guided explorations, which are two examples of a very rich set of possibilities. What to choose in a classroom situation? The teacher's decision has to be contextualized and depends on a lot of issues: the time constraints, the curriculum, the students' qualifications and so on.

This last issue is related to teacher education. The teacher's knowledge in order to design and to manage in the mathematics classroom this kind of activities is complex (e. g. Bartolini Bussi & Maschietto, in press; Maschietto & Bartolini Bussi, invited paper under review) and does not fit in the space of this paper. A systemic approach to teacher education is now in the foreground in the literature on didactics of mathematics.

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[i] The Laboratory of Mathematical Machines at the Department of Mathematics of Modena is a well known research centre for the teaching and learning of mathematics by means of artefacts (Maschietto, 2005). The name comes from the most important collection of the Laboratory, containing more than two hundred working reconstructions (based on the original sources) of mathematical artefacts taken from the history of geometry. Briefly, a mathematical machine (in the geometry subject area) is a tool that forces a point to follow a trajectory or to be transformed according to a given law.