Taiwanese Undergraduates' Performance Constructing Proofs and Generating Counterexamples in Differentiation

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Abstract

Recently, a growing number of studies in the United States have provided evidence that students have difficulty with proofs in advanced mathematics courses (Moore, 1994; Weber, 2001). Few research studies, however, have focused on undergraduates' abilities to produce proofs and counterexamples in differentiation and, more specifically, on Taiwanese undergraduates' abilities. In this study, we examine Taiwanese undergraduates' performance constructing proofs and generating counterexamples in differentiation. While not designed as a comparative study, our analysis provides results that may be compared with existing empirical research in the United States. Such comparisons can provide insight into performance differences among undergraduate mathematics students—insights that may be particularly meaningful given that Taiwanese elementary and secondary students score consistently high on international mathematical achievement tests. More importantly, our study has broader implications for instructors who would like to improve undergraduates' proof performance in advanced mathematics courses.

Introduction

Proof is a crucial ability in advanced mathematical thinking, because it helps mathematicians to see whether and why the proposition or theorem is true. Accordingly, undergraduates in advanced mathematics courses are expected to master the skills required to both construct proofs and generate counterexamples. Existing empirical studies in the United States, however, have shown that many undergraduate students have difficulty with proof (Harel & Sowder, 1998), particularly with constructing formal proofs in advanced mathematics courses, such as introductory group theory and other introduction courses to higher mathematics (Moore, 1994) and abstract algebra (Weber, 2001). Yet few studies have focused specifically on undergraduates' abilities to produce proofs and counterexamples in differentiation, and even fewer on such performance of Taiwanese undergraduates.

In this study, we examine thirty-six Taiwanese undergraduate mathematics majors' performance constructing proofs and generating counterexamples in differentiation—an important area within the undergraduate mathematics curriculum. This study was guided by the following two research questions: (1) How well do Taiwanese undergraduate mathematics majors[1] construct proofs and generate counterexamples in the domain of differentiation? (2) What errors appear in student-generated proofs or counterexamples? The data consist of students' responses to a written assessment in which they either constructed proofs for statements they considered to be true or generated counterexamples for statements they considered to be false.

While this study is not designed as a comparative study, it does provide results that can be considered in relation to existing empirical studies in the United States. Given the successful performance of Taiwanese elementary and secondary students on international mathematical achievement tests, such comparisons may provide insight not only into performance differences among undergraduate mathematics students, but more importantly into the design of curriculum and instruction that may lead to improvements in undergraduate students' performance producing proofs and counterexamples in advanced mathematics courses.

Theoretical Perspective

In this section, we discuss the relationship between concept definitions and either proofs or counterexamples. Vinner and colleagues (Tall & Vinner,1981; Vinner, 1983; Vinner & Dreyfus, 1989) characterize concept definition as a *formal definition that can be written or spoken in mathematical language*. According to Chin and Tall (2000), students often develop an understanding of concepts informally before learning their formal definitions. Moreover, informal concept definitions are essentially informal descriptions of syntactic knowledge with students' own language, and are often paired with a partially correct understanding of the formal definition. From Tall's (1989) point of view, a mathematical proof requires that "*clearly formulated definitions and statements*" or "*agreed procedures*" are used to "*deduce the truth of one statement from another*" (p. 5). Moore (1994), however, notes that students often lack

[1] Further references to the Taiwanese undergraduate mathematics majors will be abbreviated to the Taiwanese undergraduates for the sake of simplicity.

a relevant understanding of concept definitions and that they often do not know how to apply the concept definitions in the specific domain of writing proofs. Zaslavsky and Peled (1996) found that mathematics teachers, who received an undergraduate degree in mathematics, and student teachers, who have completed several advanced mathematical courses, also have difficulties generating counterexamples due to their lack of conceptual understanding. Taken together, proofs and counterexamples involve checking true and false statements. Without adequate understanding of the content of informal and formal definitions, it is likely difficult for students to determine the truth or falsity of a statement and to produce a proof if the statement is deemed true or a counterexample if deemed false. In this study, we take the view of concept definitions regarded as informal and formal definitions to explore Taiwanese undergraduates' proof performance in the domain of differentiation.

Methods

Participants in this study were selected by convenience sampling as participants were contacted by colleagues of the researchers, and were selected on the basis of their willingness to participate in the study. Participants included thirty-six Taiwanese undergraduates enrolled in Advanced Calculus I in Fall 2007 at a national university in Taiwan. Differentiation was addressed in a previous calculus course, thus all of the students participating had some relevant domain knowledge. This study did not target participants based on age, gender, or other characteristics, but the course enrollment gender distribution provided more men than women as participants.

The instrument was written in English because English is used in advanced calculus courses at the university in Taiwan. The instrument, which was comprised of five mathematical statements modified from textbooks and entrance examinations, was designed to provide a measure of students' concepts of differentiation as listed in Table 1.

Table 1

Five Propositions Used in This Study

Problem	Mathematical statement	T or F
1.	Let f be a function defined on a set of numbers S, and let $a \in S$. If f is continuous at a, then f is differentiable at a.	False
2.	Let f be a real-valued differentiable function defined on $[a, b]$. If $f'(a) < \lambda < f'(b)$, then $\lambda = f'(z)$ for some z in (a, b).	True
3.	Let f be strictly increasing on $[a, b]$, then $f' > 0$ on $[a, b]$.	False
4.	Let f and g be functions defined on a set of numbers S, and let $a \in S$. Neither f nor g is differentiable at a, then $f + g$ is not differentiable at a.	False
5.	Let f be a function defined on a set of numbers S, and $a \in S$. If f is differentiable at a, then f is continuous at a.	True

The mathematical statements were designed to (a) reflect understanding of differentiation, (b) be representative of basic types of proofs and counterexamples, and (c) be completed by each participant in 30 minutes. The instrument was finalized after pilot testing with Taiwanese undergraduates and graduate students with a major in mathematics.

Data were gathered on concept definitions through students' proofs, correctness of proofs and counterexamples. Codes for the data are described in Table 2, Table 3, and Table 4.

Table 2

Descriptions of Codes for Concept Definitions of Proofs

Code	Description
No response	Left blank
No basis of definitions	No relevant syntactic knowledge presented
Informal definitions	An informal description of syntactic knowledge with students' own language and with partially correct understandings
Formal definitions	An essentially correct formal definition in full detail using mathematical language
Incorrect formal definitions	A partially correct description using formal definitions in mathematical language

Table 3

Descriptions of Codes for Correctness of Proof

Code	Description
No response	Left blank
Counterexample	Giving an incorrect counterexample instead of a proof
No basis for constructing a proof	No relevant syntactic knowledge presented like a guess
Not proof but relevant information presented	Only narrating relevant syntactic knowledge presented
Result achieved with some reasoning omitted	Almost presenting a complete proof but making minor errors
Completeness	A complete proof

Table 4

Descriptions of Codes for Correctness of Counterexample

Code	Description
No response	Left blank
Proof	Giving an incorrect proof instead of a counterexample
No basis for generating a counterexample	No relevant syntactic knowledge presented like a guess
Not counterexample but relevant information presented	Only narrating relevant syntactic knowledge presented
Result achieved with some reasoning omitted	Almost presenting a complete counterexample but making minor errors
Completeness	A complete counterexample

The manifested errors in the students' attempts to construct proofs and to generate counterexamples were investigated by analyzing student's written work. To check the reliability of the coding, the investigators and an independent coder worked separately, and ten students' responses were selected randomly to be coded by an independent coder. Agreement between the investigators and the independent coder was 82% for coding the concept definitions of proof, 79% for coding the correctness of proof, and 84% for coding the correctness of counterexample.

Results

We first focus on the quantitative data of Taiwanese undergraduates' responses to a written instrument in differentiation, and then on what errors appeared in students' written work of proofs and counterexamples.

Quantitative Data of Taiwanese Undergraduates' Responses

Table 5, which displays the types of concept definitions used by students when writing proofs for true statements in differentiation, shows none of the students used formal definitions to construct proofs.

Table 5

Types of Students' Concept Definitions in Writing Proofs

Problem number	No response		No basis of definitions		Informal definitions		Formal definitions		Incorrect formal definitions	
	n	%	n	%	n	%	n	%	n	%
2.	8	22	19	53	5	14	0	0	4	11
5.	5	14	14	39	14	39	0	0	3	8

Illustrating students' performance in constructing proofs for true statements in differentiation, Table 6 shows none of the students provided a complete proof for each true mathematical statement.

Table 6

Students' Performance Constructing Proofs

Problem number	No response		Counter-example		No basis		Relevant inform, not proof		Result achieved, reasoning omitted		Complete	
	n	%	n	%	n	%	n	%	n	%	n	%
2.	8	22	6	17	13	36	9	25	0	0	0	0
5.	5	14	4	11	10	28	14	39	3	8	0	0

As evidence of students' performance generating counterexamples for false statements in differentiation, Table 7 indicates twenty-one and five students generated complete counterexamples for the Problem 1 and 4, respectively.

Table7

Students' Performance Generating Counterexamples

Problem number	No response		Proof		No basis		Relevant inform, not counter-example		Result achieved, reasoning omitted		Complete	
	n	%	n	%	n	%	n	%	n	%	n	%
1.	1	3	7	19	2	6	4	11	1	3	21	58
3.	1	3	31	86	1	3	3	8	0	0	0	0
4.	9	25	9	25	7	19	4	11	2	6	5	14

Errors Manifested in Students' Written Work

In general, many students were incapable of applying formal mathematical definitions to construct formal proofs for Problem 2 and 5, because they only presented no basis of definitions. With respect to constructing formal proofs, some students (25% for Problem 2 and 39% for Problem 5) only narrated relevant syntactic knowledge presented with their own language and with partially correct understandings of concept definitions in differentiation. Perhaps most surprising, none of the students were able to construct complete formal proofs in this study although they provided relevant mathematical knowledge with a partial understanding of concept definitions in differentiation. Therefore, participants were unable to apply formal mathematical definitions to write formal proofs with mathematical language. With respect to generating counterexamples, several students (19% for Problem 1, 86% for Problem 3, and 25% for Problem 4) believed the false mathematical statements to be true and attempted to provide a proof.

Discussion

Even though all participants had some relevant domain knowledge in differentiation having studied in a previous calculus course, writing a correct proof requires an understanding of the relevant concept definition, and the majority of the participants failed to express such an understanding. Additionally, several of them tried to produce a counterexample for a correct statement, and many of them were not able to produce complete correct proofs for the true mathematical statements because they seemed to lack the understanding of the mathematical language needed when writing formal proofs. Finally, the percentages of students able to provide complete counterexamples for Problem 1 (58%) and 4 (14%) were much higher than the percentage able to produce complete proofs for Problem 2 (0%) and 5 (0%). Overall, several participants struggled with figuring out which mathematical statement was correct or incorrect, and many of them struggled with writing formal proofs, demonstrating partial correct understandings of concept definitions.

Conclusions

This study asked participants to construct proofs for statements that they believed to be true and to generate counterexamples for statements they believed to be false. Although participants only responded to five mathematical statements, these results provide suggestive evidence regarding Taiwanese undergraduates' understanding of concept definitions in differentiation. This study also confirms findings from other studies in the United States (Moore, 1994; Weber, 2001); however, the results are somewhat unexpected given that the successful performance of Taiwanese elementary and secondary students on international mathematical achievement tests might lead one to expect the proof performance of such students to be better. Given the mathematical backgrounds of the participants, it is surprising that they still had considerable difficulty generating proofs and counterexamples. Such difficulty was unanticipated in part because of so little existing literature on undergraduate Taiwanese students and proofs. Additionally, the comparisons between this study and existing empirical studies in the United States (Moore, 1994; Rin, 1983) show that Taiwanese undergraduates have similar difficulties to American undergraduates when writing proofs. Such comparisons allowed by this research suggest that Taiwanese undergraduates encounter similar challenges as their American counterparts.

To develop undergraduates' understanding of concept definitions, mathematics instructors may consider a stronger emphasis on formal definitions. To assist undergraduates in writing proofs and counterexamples, mathematics instructors should pay more attention to students' written work. Although changing the pedagogies of mathematics instructors is a difficult process, such an enhancement may be a necessary condition for enabling students' understanding of concept definitions and performance in writing proofs and counterexamples.

Currently, few research studies have specifically focused on undergraduates' abilities to produce proofs and counterexamples, especially in Taiwan. To gain more insight into the relationship between undergraduates' concept definitions and either proofs or counterexamples, further research in Taiwan as well as in various countries is needed. This research could involve designing more mathematical statements and conducting intensive interviews with students to understand their perspectives. We hope this paper highlights the need to call more attention to empowering instructors in their teaching and undergraduates in their learning to write complete proofs and counterexamples with a deep understanding of concept definitions in mathematics courses in Taiwan and the rest of the world.

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