PROOF IN ROMANIAN HIGH SCHOOL INTRODUCTORY ANALYSIS TEXTBOOKS – A HISTORICAL OVERVIEW

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Abstract

We analyzed five introductory textbooks to mathematical analysis, covering a period of 40 years, in order to identify the views on proof in each of them. We were interested in identifying the role given to proof over time, the way of doing and writing down a proof (as promoted by the textbook) and the types of problems that required proofs. Therefore, our analysis comprises the study of the language employed in proof, types of explanations, level of rigor, the role assigned to proof and problems types given in the textbook. As far the language is concerned we focus on three aspects: the use of everyday words in explaining concepts, multiple definitions and level of formalism. By types of explanations we refer to the use of visual and experimental arguments along with typical examples and counterexamples. By the level of rigor we mean the measure to which the proof contains all the arguments for the statements that contains.

Our main conclusion is that there was a slowly increasing emphasis on proof until the 90s and a considerable decay afterwards in favor of an algorithmic treatment of introductory analysis. We briefly expose the reasons for such a turn and its foreseeable consequences.

Research oriented; Theme: Curriculum and textbook aspect.

Introduction

In the last 20 years studies related to mathematical proof gained terrain. There was an important change on the perception of the role that mathematical proof plays not only in mathematics, but also in teaching and textbooks (Hanna – 1996, 2001, 2007; Rav, 1999; de Villiers, 1999 – just to mention few). Hanna (2001) enumerates eight functions of mathematical proof based on literature review: 1) verification (concerned with the truth of a statement); 2) explanation (insight into why it is true); 3) systematization (organization into a deductive system of axioms); 4) discovery (obtaining new results); 5) communication (of knowledge); 6) construction of an empirical theory; 7) exploration (of definitions or consequences of assumptions); 8) incorporation of a well-known fact into a new framework. Of all these functions, Hanna considers that explanation is the most important one in education. Rav (1999, pp.13) underlines another aspect of proof as he writes “proofs are the mathematician’s way to display the mathematical machinery for solving problems and to justify that a proposed solution to a problem is indeed a solution”. In our analysis we focus on three particular roles of those enumerated before, namely: proof as validation, as explanation and as tool for solving problems. If these roles are important to be emphasized in teaching, it is still a question why would be proof included in textbooks. The short answer to such a question is that students view of proof is mainly gained from textbooks (based on a curriculum) and from their teachers attitude towards proof (Selden & Selden, 2007). Although the importance of proofs in textbooks was clearly stated by researchers, there are just few studies on the role of textbooks in teaching and learning proof (Hanna & de Bruyn, 1999; Nordstrom & Lofwall, 2005; Grenier, 2000). As Hanna & de Bruyn (1999)

stated: “In particular, there has been almost no examination of the actual occurrence in mathematics textbooks of proofs, discussions of proof, and exercises requiring the construction of proofs” (pp. 180).
Our aim is to make this type of analysis on textbooks that were used in different periods of time. In Romania introductory course to mathematical analysis begins in the third grade of high school (17-18 years students) and comprises the fundamental concepts such as sequence, limit, convergence, function continuity, derivability, etc. The overall organization of the third year high school curriculum barely changed since the 60’s, however there are differences between the newly edited textbooks in terms of approach.

Methodology

We analyzed five textbooks, edited in 1965, 1978, 1984, 2006 and 2007. Before 1989, in Romania, the curriculum was revised every six-eight years, leading to the creation of new textbooks. The same textbook had several editions since had to cover a huge range of users. It has to be said that the same textbook was used by all students attending the same profile class, independently of the city or high school. After 1990, changes in curriculum became extremely frequent and official textbooks are selected by a commission from a set of proposals. Hence, for the same profile there are available different textbooks. We chose for the analysis textbooks from two consecutive years from the same editorial group. Our focus is on the section that introduces sequences, limit and convergence. We propose a qualitative and comparative analysis of the textbooks by attending specific issues related to proof and the particularities of mathematical analysis concepts.

With these concepts students enter the field of advanced mathematical thinking and for the first time they are confronted with notions that can’t be fully understood just from their more basic constituent concepts. The complexity of these concepts difficults students’ understanding (Nesher & Kilpatrick, 1990). In order to be able to operate with them, along other issues, the student needs to develop an intuition specific for the limit process. Singer & Voica (2003) argue that the human mind has a procedural and a topological dimension; complementary by their nature they allow us to describe our environment. It seems that younger students have a process view of infinity that in time is complemented with a topological understanding. In order to turn on these dimensions of the understanding, textbooks appeal to shifts between geometric intuition and algebraic formalism. Similarly, Vinner (1983) speaks about the “concept image” as the visual representation of the concept (if any) or as the collection of impressions or experiences. In order to truly understand a concept one has to swing between concept definition and concept image. Therefore, the real question is how to ponder these approaches when introducing new concepts? On other hand, examples have a special place in a mathematician’s “toolbox” by the uses that are given to it in proving tasks. These were synthesized by Alcock (2004) as: understanding the statement, generating an argument and checking an argument. Therefore, there is a special interest in seeing what types of examples are used in the five textbooks along with the textbook authors’ options between an intuitive and formal approach.

Somehow related to the above described issue is the use of everyday language in explaining concepts. Sierpinska (1985) identified five epistemological obstacles in the understanding of the limit concept. Language can bias our perception of what is the case in a given situation, by transferring the significance of everyday words into a mathematical context. Expressions like “tend to” or “get closer and closer” induce an incorrect interpretation

when used related to some concepts. We compare the occurrences of such words in the five textbooks.
Definition of a concept plays a major role in proof by accomplishing three roles (Moore, 1994): it provides the language to writing proofs; suggests the sequence of individual steps and the justification of each in a proof and, finally, reveal the logical structure of proof. However, in order to accomplish these functions, the formulation of the definition has to be given in such way that to allow the translation into practical steps. We shall describe the changes occurred in time in the formulation of the definition of some concepts. As third point, we take a look at the use of mathematical formalism. Sierpinska (1985) found that the use of quantifiers to describe relations between terms or concepts make difficult for students to grasp the meaning of the statement. Even if when mathematical formalism promotes communication it can be a handicap if used at extreme in introductory courses.

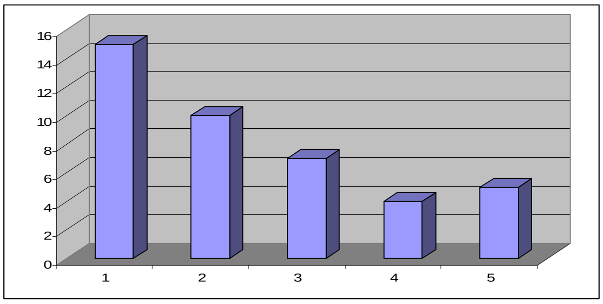
A proof, in order to be accepted, has to have a justification of the steps involved in it. However, especially in introductory textbooks to mathematical analysis, some aspects are assumed known or obsolete and therefore omitted. Such situation occurs frequently when demonstrating some convergence property on R; the topological and order structure of R is taken for granted and omitted even if it is obvious that the same argument could not work in lack of it. We examined the textbooks proofs from this point of view.

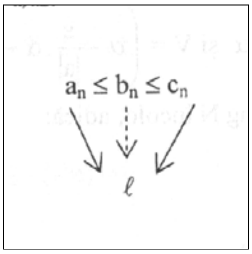
At last, our focus was directed towards problems presented at the end of a section. Some of them are meant to promote understanding and fixation of the concepts and involve proof. We counted the problems that asked for a proof having that specific explanation power mentioned by Hanna. She cites Steiner (1978) who says that an explanatory proof will make "…reference to a characterizing property of an entity or structure mentioned in the theorem, such that from the proof it is evident that the results depend on the property" (pp. 143). In the present analysis we concentrated on problems that involved proof by definition. In continuation we outline our results.

Results

Types of explanations

Textbooks contain graphical or scheme representations in order to aid intuition in the formation of concept images (in our case, concepts like neighborhood, monotony, bounded, and convergence). We found that the number of such representations (as corresponding to the chapter about sequences) diminished in time, suggesting the existence of the tendency to pass from intuitive to more formal, algebraic introduction of the concepts (figure 1a,b). There are two more remarks to be made. First, the number of themes treated in this chapter drastically decreased in the last two textbooks. Second, in the textbook from 1986 figures correspond to particular sequences, given by an expression or having some physical interpretations, meanwhile in the others is used for generic sequences.





1a. 1b.

Figure 1a. The number of graphical representations used in the textbooks in sequence chapter (1-1965, 2- 1978, 3 – 1984, 4 – 2006, 5 - 2007);
1b. Geometrical representation of the major-minor term rule

It worth mention that in the introduction of the 1965's textbook the authors explicitly stated their intuitive approach by saying "we opted (…) to treat sequence limits in terms of neighborhoods, by considering that in such way there are possible to give definitions and promote reasoning closer to intuition, than by the means of ε as in classical analysis". (Dinculeanu & Radu, 1965). Textbooks from 1978 and 1984 introduce the ε-N characterization theorem of the limit and use it in proving the existence of the limit. In the 2006 and 2007 textbooks, however, the theorem is omitted and instead proving uses the neighborhood definition along with the definition of the neighborhood itself.
Also, we remark that the first two textbooks give a shorter form of some theorems (emphasized by enclosing it in a centered rectangle), ignoring the conditions under which these are true (for example, the theorem regarding the convergence of the sum of two convergent sequences).

Typical examples given after the definition of a concept help to construct concept image. In table 1 we show what examples were used in the textbooks to illustrate the concept of non-monotone sequence.

Textbook	*Main examples of non-monotone sequences*
1965	1, - 2, 3, - 4, …
1978	1, - 2, 3, - 4, …; 1, 0, 1, 0, 1, …
1984	$((-2)^n)_{n\in N}$; $\left(\frac{1+(-1)^n}{2}\right)_{n\in N}$
2006, 2007	Do not appear (as an explicit example)!

Table 1. Typical examples for non-monotone sequences in the five textbooks

We can see that there is also an evolution in considering typical examples. In the first two textbooks, which consider intuition as a main tool to understand mathematical analysis, the examples are much more concrete, comparing with the ones given in the third textbook (published in 1984). On contrary, the last two textbooks completely ignore the issue of examples (at least, for some concepts).

Language

Everyday terms are used in all textbooks, especially when talking about an introductory example, before the formal definition. Expressions like "getting closer and closer", "they accumulate" or "as the neighborhood is getting smaller it contains less terms" are recurrent in these textbooks. However, only the textbooks from 1978 and 1984 state explicitly that such an intuition must be followed by a formal verification.

As about definitions, an interesting detail was detected. We give the definitions for sequence limit in table 2.

Textbook	Definition of sequence limit
1965	The number *a* is the limit of a sequence if outside of all neighborhoods of *a* there are at most a finite number of terms.
1978	The number *a* is the limit of a sequence if any neighborhood of *a* contains all the terms except a finite number.
1984, 2006, 2007	The number *a* is the limit of a sequence if any neighborhood of *a* contains all the terms from an index on.

Table 2. Definitions for the limit of a sequence as given in different textbooks

Although in these formulations is the very same concept defined, from practical point of view the last one it is easier to apply. If one would like to prove the limit with definition given in 1965, it would need to reformulate it. However, students are knowingly bad at manipulating logical statements with quantifiers. From the point of view of formal expression, the textbook of 1984 excels. All statements are given as short as possible, in compact formal form. Proving goes on by implications (with arguments whenever necessary) in a strictly deductive manner by rewriting the formal expressions. Earlier textbooks combined verbal explications with some formalism; later textbooks give very few proofs and those mostly in terms of previously known results (like based on an earlier announced theorem). The high level of formalism can prevent students to see the proof as a model to be used in problem solving since some steps could not be straightforward for them all, however the complete lack (or a very reduced number) of proofs based on definitions will let students without a tool.

Rigor comes close to formalism. The 1984^{th} textbook has a high level of rigor, because the given proofs contain all the arguments for the statements that appear in it. (In addition, this textbook starts with an axiomatic construction of the set of real numbers). For example, Weierstrass' theorem proof is not presented in the two previous textbooks; instead an explanation is given, that is, such proof is beyond the textbooks frame. The theorem is demonstrated with the entire required rigor in the 1984 textbook, but it is just enounced in the last two ones. One could ask if it is really necessary for the students to know about the theorems' proof. We consider that once all the necessary element for understanding the proof are given, the proof should be presented in order to give students the opportunity to understand and learn from its structure. Recent textbooks are interested in an algorithmic presentation of sequences and limit calculus, with a special accent on practical theorems that allow computing instead of proving.

Problems requiring explanation proof show the same phenomena of increasing in proportion until 1984 textbooks and drastically diminishing afterwards. The number of overall problems is relatively reduced in textbooks before 1984 and theoretical or proving problems are scarce. The textbook of 1986 instead contains a lot of problems that needs strong theoretical backgrounds: like proving by definition or open ended questions regarding sequences with given properties, but also constructions problems (of specific sequences). In later textbooks, exercises dominate. No theoretical issues are explored, no open ended questions posed. Instead the focus is on algorithmic performance, by repetition of “problem solving recipes”.

In overall, it can be said that since the 1984 textbook there was a constant overview of the role of proof in mathematics, seemingly putting more accent on routine problem solving performance instead on deepening student’s understanding of analysis.

Conclusions

The analysis highlighted the tendency of ignoring proof and its role in deepening mathematical understanding. Unfortunately, there are other factors that favor such an approach, like the introduction of national tests in which algorithmic problems dominate admission exams to University level education based on multiple choice tests with numerical problems or the abolition of admission exams. Hanna (2000) mentions three specific factors that lead to the decline of proof in the curriculum: 1) the idea that proof need to be taught only for those who will pursuit education at higher levels; 2) the view that favors heuristic techniques instead of proof when it comes to developing skills in reasoning and justification; 3) the idea that deductive proof can be abandoned in favor of a dynamic visual approach to mathematical justification. Mathematics educators insist that proof has to be an essential part of the curriculum due to its multiple roles in mathematics education. Also, many research papers speak about the negative consequences, on one hand, for prospective University students, of not explicitly teaching proving techniques in high schools and, on other hand, of not illustrating the different roles that proof can take in math education for prospective teachers. In case of Romania, the effects of curriculum modifications as exposed above can already be traced in the decreased level of preparation of University first year students. The present analysis also has the purpose of focusing attention on this particular issue by formulating aspects that should be considered in the textbooks preparation.

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