REASONING AND PROOF IN CLASSRROM (9TH GRADE)

Margarida Rodrigues – Escola Superior de Educação de Setúbal, Portugal –

mtrodrigues@ese.ips.pt

ABSTRACT. This paper is research oriented and pretends to contribute toward giving empirical evidence about how students develop their reasoning and how they achieved to a proof construction in school context. Its main theme is epistemology. It describes the way in which four students in 9th Grade explored a task related with the discovery of symmetry axes in various geometric figures. The proof constructed by students had essentially an explaining function and it was related with the symmetry axes of regular polygons. The teacher's role in meaning negotiation of the proof and its need is described through illustrative episodes. The paper presents part of a study which purpose is to analyse the nature of mathematical proof in classroom, its role and the nature of the relationship between the construction of a proof and the social interactions. Assuming a social perspective, attention is focussed on the social construction of knowledge and on the structuring resources that shape mathematical experience. The study's methodology has an interpretative nature. One outcome of the study discussed here is that students develop first a practical understanding with no awareness of the reasons founding mathematical statements and after a theoretical one leading them to a proof elaboration.

Theoretical issues

The study's framework is rooted in the theoretical frame of activity theory in the line of Vygotsky and Leont'ev. Drawing on a vygotskian approach, Wertsch (1991) uses the bakhtinian construct of 'voice' to emphasise the social origins of individual mental functioning. The process whereby one voice speaks through another voice in a social language is termed 'ventriloquation' by Bakhtin (1981). So there is a certain interference of one voice on another accompanied by a partial and correlative subordination of the last one.

Mathematics learning is seen as a situated phenomenon (Brown, Collins, & Duguid, 1988; Lave, 1988; Wenger, 1998). As the school context plays a fundamental role, it is not possible to separate activity, people acting—and respective interactions—and the artefacts that mediate that action. All those dimensions are intrinsically interwoven. The study draws also from embodied cognition perspective (Lakoff e Núñez, 2000) assuming that mathematical concepts are structured by the nature of our bodies and the particular way we function in the world.

Knowledge is not independent of the situation in which it is produced. If situation is structuring of cognition, then we can assume also that knowledge and activity are inseparable and mutually constitutive. The centrality of activity in cognition constitutes the base for study theoretical background. It is the mutual interaction between acting and knowing that shapes one another reciprocally (Rodrigues, 1997). Cognition includes the use of representations but is not based on them. The emphasis falls on the notion of action and the relationship between the subject and the world is redimensioned: the subject and the object, that is, the interpreter and the interpreted define one another simultaneously and they are correlatives (Varela, 1988/s.d.); they are not independent nor are separate entities as assumed by the rationalistic perspective.

The proof is inherent to the nature of mathematics as a science. So the study focus the mathematics philosophy discussing questions as the nature of mathematical objects, the relationship between the experimental reality, the natural and human world and the mathematics, how is seen the truth and what can rely it. The study discusses the epistemological status of proof, assuming mathematics as a human and social construction, but non-arbitral. It is this non-arbitrary that explains the parallelism between the physical reality and the mathematical one (Hersh, 1997). In the line of Ernest (1993), the mathematical knowledge develops through conjectures and refutations (Lakatos, 1994) and relies on linguistic knowledge, conventions and rules.

Aims and Methodology of the Study

The methodology adopted has an interpretative nature because it is adequate to the aims of the study that are to examine: (1) the role of the proof in a classroom in various aspects such as (a) mathematical understanding, (b) validation of mathematical knowledge, and (c) mathematical communication; and (2) the relationship between the construction of a proof and the social interactions occurring in a classroom. The social interactions are analysed drawing on a hermeneutic conception of activity and context (Winograd & Flores, 1993) and on a social theory of learning (Wenger, 1998) by questioning (a) students group dynamics and (b) the power relations in the students group. In contexts of work using inquiry pedagogy, the study pretends answering these questions: (1) what is the nature of the mathematical proof and what are their characteristics in a 9th grade classroom?; (2) what is the role of the proof in a mathematics classroom?; and (3) what is the nature of the relationship between the construction of a proof and the social interactions?

The analysis unit was students' mathematical activity. Through the analysis of scholar mathematics practice, I have pretended to understand how students reason in this practice and how is negotiated the meaning of proof, studying the phenomenon in its natural setting—mathematics classroom. For that reason, I was attentive on all aspects concerned with students practice: their utterances, their acting, their facial expressions, the mediator resources.

The data were collected in a public school in two classes of 9th grade, during two years, with the same teacher. In each class, four students were videotaped. The researcher assumed the role of participant observer, having observed and participated in all mathematical activities of each class during two periods of the first year and 16 lessons of the second year in which the inquiry tasks were explored. The data were collected by: (a) video record of the mathematical activities of the students, (b) audio record of students' dialogues, (c) field notes done by the researcher, (d) video record of the students and teacher semi-structured interviews, and (e) documental analysis of the work done by the students and of the video and audio records.

The video records assume a great importance in data analysis because they allow doing observations of behaviour procedures (so many as necessary) after occurring. They enable also to capture details that could be ignored by the direct observation of the researcher in classroom on the occurrence moment.

Discussion of some results

In order to present details data in this paper, I chose just episodes occurred with a group of four students (of the second year of data collecting) and related with a single task: the discovery of the number of symmetry axes of various geometric figures, using mirrors.

Structuring resources

Mirrors and the drawing of symmetry axes were structuring resources since they shaped the process of conjecturing and constructing a proof. According to Lave (1988, pp. 97-8), "such resources are to be found not only in the memory of the person-acting but in activity, in relation with the setting, taking shape at the intersection of multiple realities, produced in conflict and creating value". Students used mirrors, first to discover the localization of triangle and hexagon symmetry axes. After, they put the mirrors on the correct localization of symmetry axes, without drawing them. They had visualized the symmetry axes: they didn't search the right localization of the mirror. Here, the act of putting the mirror just had confirmed what students had visualized before. Since the teacher has demanded to draw the axes, students drew them using the ruler, dispensing the mirror. They drew the axes where they saw them through their mental images. The reified character of drawing axes was very important to the conjecture formulation (enabling counting and counting again) and to the production of the proof (enabling the observation by where cross the axes). Wenger (1998, p. 58) defines reification as “the process of giving form to our experience by producing objects that congeal this experience into ‘thingness’”. Reification shapes the experience: drawn axes changed students experience by focusing their attention and enabling higher levels of understanding. This is the power of reification, according to Wenger (1998, p. 61): “its succinctness, its portability, its potential physical persistence, its focusing effect”. After drawing the axes, students used the mirror twice as a confirming tool. Sara used it to verify if Ricardo had drawn the pentagon’s axes correctly. Ricardo used it to prove to Sara that she was wrong when she drew on the heptagon an axe linking two vertices. First Sara began stammering out a few words pretending to defend its construction using a mathematical argument—“If I put like this it will measure”—but Ricardo didn’t let her finish—“Mirror!”. He used this artefact as an empirical argument to prove the incorrect localization and to show the correct one. So the mirror validated Ricardo’s assertion, arbitrating the discussion between them.

Conjecturing

The observation of the pattern related with the same number of sides and symmetry axes of the first regular polygons appeared in the table—triangle and square—led pupils to generalize the pattern to the other regular polygons and to formulate a conjecture: “I’m seeing that the sides’ number and the axes’ number will be always equal.”—said Sara. Three episodes show that students do not regard this conjecture as a suspect proposition: they believe it is true.

Episode 1. This conjecture will be rejected, for a moment, by Sara and Maria, with a lot of resistance, when they judged facing a counter-example. The hexagon was represented in the paper immediately on the right of the square in an unusual position (Figure 1) and they assumed to be the pentagon, following the same order of the table.

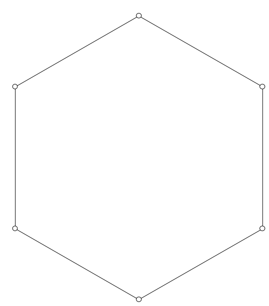

Figure 1. Hexagon position

Sara and Maria reveal difficulty at recognizing the figure in a different position, that is to say, at using the capacity of perceptual constancy (Del Grande, 1990). Sara, perplexed, counted and counted six axes, for a total of five times, where she expected to count just five axes, according to the conjecture formulated. Then, defrauded, Sara wrote the number six, in the table, below the number five respecting to the sides. After a while, Ricardo pointed to the hexagon affirming that it had six sides. And the two girls corrected what they had written before. This episode reveals the conviction on the conjecture. The two pupils had just yielded to the evidence of six axes in a polygon that they thought to have five sides after counting the axes a lot of times. In spite of the fact that students believe in the conjecture, probably they would not reject the contradiction by excluding it as a special case, which Lakatos (1994) named ‘monster-barring’. However it is impossible to know how their reasoning would evolve in this case because Ricardo’s intervention had dissolved the contradiction.

Episode 2. Ricardo had drawn five axes on the heptagon. Bernardo refuted, after counting them:

B- Silva, this is wrong, Silva. Seven.
R- Seven or five?
B- It's missing to draw yet…
R- Ah! Give it to me! (*he draws two axes more on the heptagon*).

Bernardo did not argument referring the figure symmetry nor used the mirror to confirm or to indicate the missing axes. He just saw that his counting did not coincide with the conjecture that postulates that a polygon of seven sides must have seven axes. It is a refutation based on the power of the conviction that the conjecture is true.

Episode 3. Sara drew some axes on the octagon. Then, she counted them and continued drawing the other axes until having a total of eight axes. The conjecture guides her work helping her seeing where they are located, since she assumes that must draw eight axes.

So, when exploring the task, students generalize to *n* sides the pattern observed in concrete polygons, assuming that it is true. They wrote "The conclusions we can achieve that's [sic] the sides' number of the regular polygon is always equal to the symmetry axes' number" and in the table they wrote *n* axes below *n* sides. The ongoing process of sequence of regular polygons deals with potential infinity (Lakoff & Núñez, 2000) because it is an uncompleted sequence since we cannot construct the last polygon corresponding to the final result. However, when students complete the table, they disconnect from the concrete polygons and they think only on the sequence of natural numbers (starting in 3) respecting to the sides' number and to the symmetry axes' number. In this sequence, *n* is conceived by students as a variable assuming any integer value larger than 2, as well as ∞, taken as a number in an enumeration, taken as its extremity—the actual infinity, conceptualized metaphorically by Basic Metaphor of Infinity (Lakoff & Núñez, 2000) as a final and unique resultant state (the 'natural' and unique ∞ larger than any finite natural and beyond all of them). Here the Basic Metaphor of Infinity is applied to the special case of enumeration through the addition of the metaphorical completeness. This conjecture starts from the potential infinity of an unending sequence without final polygon to establish a relationship between the sides' number and the symmetry axes' number of regular polygons. So when students do this generalization, they conceptualize the ongoing process of the sequence of natural numbers in terms of a completed process, that is to say, they produce the concept of actual infinity.

Also when conjecturing about the infinity of symmetry axes existing in a circle, students apply the Basic Metaphor of Infinity in this special case. We can see a brief extract of their dialogue:

S- Look, the circle... It has axes… It has, it has infinite symmetry axes…
R- The circle?? The circle?? Saaaara!!! Of course it has infinite. The circle is all round.
(…)
B- I don't know if it is infinite.

Why would Bernardo doubt about the infinity of circle's symmetry axes? This is an abstract idea. Even if Bernardo would put the mirror on the circle, he would put it a finite number of times. Even if Bernardo would draw the circle's symmetry axes, he would draw a finite number of axes. It would be a drawing that represents an abstract idea that can never put in practice. Considering circle's infinite symmetry axes implies to considerer also infinite points in a circumference. It is more difficult to conceptualize the infinity in a limited object with beginning and ending as a segment or a circumference than in the straight line. When Ricardo says "The circle is all round", he is conceiving that in a circle it is always possible consider an axe between any two axes and also it is always possible existing a point between any two circumference's points. The infinity of circle's symmetry axes is of numerable type (Caraça, 1998) characterized by being discrete and infinitely large. The geometric point has no dimensions and consequently it exists an infinity of points between any two circumference's points. It is a type of infinity characterized by its continuity and density (Caraça, 1998). Given any arc circle, shorter it would be, it is always possible to divide it in half and to get a shorter one. So here we enter in infinitely small things. The act of dividing in half is a mental construction that goes on unlimitedly. According to Lakoff & Núñez (2000) the aspectual system—it characterizes the structure of events as we conceptualize them—is the fundamental source of the concept of infinity. In life, nothing goes on forever. Yet we can conceptualize events as not having completions (imperfective aspect). Let us see the Basic Metaphor of Infinity applied to the circumference:

Target Domain		*Special Case*
ITERATIVE PROCESSES THAT GO ON AND ON		THE CIRCUMFERENCE
The beginning state (0)	→	The null circumference ABC_0
State (1) resulting from the initial stage of the process	→	The circumference ABC_1 where the length of arc AB_1 is D_1.
The process: From a prior intermediate state (*n*-1), produce the next state (*n*).	→	Form AB_n from AB_{n-1}, by making D_n arbitrarily shorter than D_{n-1}.
The intermediate result after that iteration of the process (the relation between *n* and *n-1*)	→	$D_n < D_{n-1}$
"The final resultant state" (actual infinity "∞")	→	**D_∞ is infinitely short. The arc AB_∞ is infinitely short.**
Entailment: The final resultant state ("∞") is unique and follows every nonfinal state.	→	**There is a unique AB_∞ (distance D_∞) that is shorter than AB_n (distance D_n) for all finite *n*.**

The Basic Metaphor of Infinity applied to the circumference

The act of dividing in half any arc, being an iterative process that goes on indefinitely and produces *n* states, is conceptualized as a complete process with a final resulting state, producing the actual infinity. Therefore the idea of infinity of circumference's points is based on cognitive mechanisms that all people use everyday as the aspectual schemas and the conceptual metaphor (Lakoff & Núñez, 2000). Despite the two different types of infinity in this conjecture—numerable and infinitely large type of axes' infinity; continuum and infinitely small type of infinity of circumference's points—we can affirm that it is the dense nature of infinity of circumference's points that leads to the other one, extending circle's symmetry axes until the infinity. In spite of the fact that these infinities are different, they are conceptually related and shape each other.

Producing a proof

So students begin their work by conjecturing. In this work phase, they believe their conjecture is true but they do not yet understand why it is true. The experimental work done with mirrors was not enough for fostering a deeper mathematical understanding. As Mason, Burton and Stacey (1980) points out, learning just occurs when students reflect on their experimental work. It is this reflective understanding that leads them doing more generalizations and constructing a narrative proof with informal characteristics: "In regular polygons: Odd– the symmetry axes cross vertex-side, and that is the reason why they have the same number of symmetry axes and of vertices; Even- the symmetry axes cross side-side and vertex-vertex and that is the reason why they have half of symmetry axes in relation to the sum of vertices with sides". For polygons with odd number of sides, students describe by where axes cross them and explain why it is the same number, referring to the vertices, assuming implicitly that in a polygon there are so many vertices as sides. For polygons with even number of sides, they describe by where axes cross them and explain why it is the same number, in spite of the fact that they do not refer explicitly the equality of numbers: this equality would be deduced from the relation reported by them—"half of symmetry axes in relation to the sum of vertices with sides"—and could be expressed algebraically as $\frac{2n}{2} = n$.

This proof has multiple functions (De Villiers, 2001; Hanna, 2000): verification, explanation and communication. However for the students the proof had a unique function: explaining why their conjecture was true. For them, the truth was yet established by conjecturing. It is for that reason that they do not feel need of deducing explicitly the equality of sides' and axes' number for polygons with even number of sides for the purpose of verifying the truth expressed by the conjecture.

The teacher discourse when interacting with group members and the questions posed by the task have a fundamental role for fostering more complex levels of student mathematical thinking. One of the questions of the task (2.c) was of fundamental importance to lead students constructing a proof: "How are the symmetry axes in relation to the vertices and sides? (By where do the axes cross?)". This question provokes students' reasoning going far away from the mere conjecture that the number is the same. I will present here some episodes of interactions between the teacher and the students' group that illustrate the meaning negotiation of the proof.

Episode 4. When students were drawing the axes and completing the table, the teacher said "Besides counting the quantity, how many are, don't forget observing how did you draw them—you linked what with what, ok?—to can answer the following questions". The teacher focuses students' attention on the way in which axes cross the polygons. It is evident here the importance of the reified character of drawn axes: the teacher does not demand the observation on the way in which they draw, at the moment they do the drawing; it is an observation after the drawing. Effectively when students draw the axes they develop a practical understanding; at the moment, they are not aware of the different behaviour of odd and even number of sides of regular polygons. It is after this practical understanding that students develop a reflective one (Heidegger, 1998; 1999) that will enable them to construct a proof for explaining why it is always the same number. However this moment is premature to teacher's intervention: they continued drawing the axes without observing consciously by where they cross. Later, when they faced with the question 2.c), students did not understand its meaning and demanded teacher's help—"Teacher, I do not understand the c)", said Sara. They did not remember the previous teacher's intervention made at a premature moment.

Episode 5. This episode begins with Sara's request of teacher's help, referred at episode 4.

S- Teacher, I do not understand the c).
T- Then, what do you not understand in c)? Go on…
Sara stammers out a few words reading the question and the colleagues read for the first time the question 2.c).
R- Well, it is here that I talked it was the mediatrix.
T- Go on… And are they all? Have they all that position that we are referring there? (*points to the chalkboard*) All axes…
R- All axes vertex-side…
T- (…) The question here is: will the symmetry axes' position be always the same? (…) (*pointing to the triangle)* But, for example, does it link the same elements, always? The same type of elements? Or does it link different elements?
B- Different elements.
T- (*pointing to the triangle again)* That it is a vertex with the opposite side's midpoint. And the three axes are of same type, isn't it? (*pointing now to the square with the ruler)* So now here, in the square.
R- (*immediately*) Side with side. Vertex with vertex.

T- And how many, how many do link opposite sides, parallels?
R- Two.
T- Two. And how many do link opposite vertices?
R- Two.
(…)
T- (*pointing to the pentagon*) Now here. Let's go to this of five.
B- It's vertex-side and side-side.
T- Is it always vertex… is it always side-side?
R- No. It's vertex-side.
T- It is always vertex-side, isn't it?
R- Teacher, then the odd is vertex-side and the even vertex-vertex, side-side.
T- Well, go on! You are thinking in a theory, isn't it? Well, let's go seeing if that theory has sense. Let's go verifying. Let's go! (*she goes away*).
Ricardo looks attentively to the paper where the polygons are represented; he counts and writes something near each polygon.

First, Ricardo was centred on the case of polygons with odd number of sides—"All axes vertex-side…"—and on the fact of symmetry axes cross perpendicularly the midpoint of sides' polygons, transporting the mediatrix's properties to symmetry axes. As we can see, the students' awareness of the different way of axes crossing in odd and even number of polygons' sides emerged only during the dialogue with the teacher when she focused their attention to the this aspect pointing to concrete polygons as the triangle, square and pentagon. It is the rising out of something appropriated before by the action of drawing the axes (the reflective understanding after the practical one, according to Heidegger, 1998; 1999). And this is why Ricardo has answered so quickly when the teacher pointed to the square. The particular instances are resources that help students generalizing. It was by looking to the concrete polygons pointed by the teacher that emerged the general way of axes crossing for all polygons with odd and even number of sides. The teacher has valued Ricardo's generalization naming it by *theory* but she did not want validate it. She asked students to verify that theory, soliciting them implicitly testing the generalization with more specializations. It was what Ricardo has done after teacher's retirement. There is a continuum movement between specialization and generalization. She also had questioned about how many axes cross opposite sides and opposite vertices in the square but it was a premature question: they were not yet thinking in the relation between the theory and the formulated conjecture. The teacher withdrew from the group at the moment she felt students understood the question's meaning and they were in a productive work of generalization needed for the proof construction. Later, the teacher will negotiate the need of a proof and its meaning: "Now try explaining, based on that, based on crossing in a way to the odd and in another way to the even, why it is always the same number". Students did not feel need for establishing any relation between Ricardo's theory and the conjecture stated before. And this relation was not asked explicitly in the formulation of the questions of the task. This intervention was fundamental to foster students' reasoning toward a proof.

In closing

Students look to the conjectures they self say as conclusions: they are certain about the truth of those conjectures and they do not feel need of submitting it to verification. The teacher has a fundamental role negotiating the need of a proof in mathematics for guarantying the validity of a statement or conjecture to the generality of cases. It implies an epistemological change: while in other disciplines as Natural Science or Physics it seems quite natural to accept a fact as a true one when it is supported by empirical evidence, in mathematics the truth is just accepted on the basis of a theoretical deduction.

Results from Rodrigues (1997) and from the present study are convergent in this point: according to the philosophical perspective of Heidegger (1998; 1999), the essence of cognition is “the pre-reflective experience of being *thrown* in a situation of acting” (Winograd & Flores, 1993, p. 97, author italics); the practical understanding is immediate and primary; students develop a theoretical and reflective understanding only after developing a practical one. The process of generalization and the process of construction of a proof are intimately associated to that theoretical and reflective understanding.

Students prefer using narrative arguments than algebraic ones. For them, the proof had the function of explaining why is true what they believe is true. According to Hanna (2000, p. 8), “in the educational domain, then, it is only natural to view proof first and foremost as explanation, and in consequence to value most highly those proofs which best help to explain”.

The group members had different forms of participating in the work. In the team videotaped only one student—Ricardo—had appropriated totally the proof. The other members group used the ventriloquation (Wertsch, 1991) incorporating, in part, his discourse given his social status more powerful. The team videotaped was analysed in the present study as a community of practice (Wenger, 1998; Wenger, McDermott e Snyder, 2002): “community is an important element because learning is a matter of belonging as well as an intellectual process, involving the heart as well the head” (Wenger et al., p. 29)

References

Bakhtin (1981). *The dialogic imagination: Four essays by M. M. Bakhtin,* M. Holquist (Ed.). Austin: University of Texas Press.

Brown, J., Collins, A., & Duguid, P. (1988). *Situated cognition and the culture of learning.* (Report Nº IRL 88 0008) Palo Alto: Institute for Research on Learning.

Caraça, B. (1998). *Conceitos fundamentais da Matemática* (2ª ed.). Lisboa: Gradiva.

De Villiers, M. D. (2001). Papel e funções da demonstração no trabalho com o Sketchpad. *Educação e Matemática, 63*, 31-6.

Del Grande, J. (1990). Spatial sense. *Arithmetic Teacher, 37*(2), 14-20.

Ernest, P. (1993). *The philosophy of mathematics education* (2ª ed.). Bristol: The Falmer Press.

Hanna, G. (2000). Proof, explanation and exploration: An overview. *Educational Studies in Mathematics*, *44*, 5-23.

Heidegger, M. (1998). *Ser e tempo – Parte II* (6ª ed.). Petrópolis: Editora Vozes.

Heidegger, M. (1999). *Ser e tempo – Parte I* (8ª ed.). Petrópolis: Editora Vozes.

Hersh, R. (1997). *What is mathematics, really?* New York: Oxford University Press.

Lakatos, I. (1994). *Proofs and refutations: The logic of mathematical discovery* (J. Worrall e E. Zahar, Eds.) (12ª ed.). Cambridge: Cambridge University Press

Lakoff, G., & Núñez, R. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being.* New York: Basic Books.

Lave, J. (1988). *Cognition in practice: Mind, mathematics and culture in everyday life.* Cambridge: Cambridge University Press.

Leont’ev, A. (1978). *Activity, consciousness and personality.* N. J. : Prentice Hall.

Mason, J., Burton, L, & Stacey, K. (1984). *Thinking mathematically.* London: Addison-Wesley Publishing Company.

Rodrigues, M. (1997). *A aprendizagem da Matemática enquanto processo de construção de significado mediada pela utilização do computador* (tese de mestrado, Universidade de Lisboa). Lisboa: Associação de Professores de Matemática.

Varela, F. (s.d.). *Conhecer as ciências cognitivas: Tendências e perspectivas.* Lisboa: Instituto Piaget. (Trabalho original em francês publicado em 1988Wenger, E. (1998). *Communities of practice: Learning, Meaning, and Identity.* Cambridge: Cambridge University Press.

Wenger, E. (1998). *Communities of practice: Learning, Meaning, and Identity.* Cambridge: Cambridge University Press.

Wenger, E., McDermott, R., & Snyder, W. (2002). *Cultivating communities of practice: A guide to managing knowledge.* Boston, Massachusetts: Harvard Business School Press.

Wertsch, J. (1991). *Voices of the mind: A sociocultural approach to mediated action.* Hempstead: Harvester Wheatsheaf.

Winograd, T., & Flores, F. (1993). *Understanding computers and cognition: A new foundation for design* (8ª ed.). Reading, Massachussets: Addison-Wesley Publishing Company.