REASONING AND PROOF IN HIGH SCHOOL TEXTBOOKS FROM THE USA

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Abstract

This paper addresses research about curriculum and textbooks. Specifically, it addresses the questions: What is the nature and extent of reasoning and proof embodied in contemporary high school mathematics textbooks in the USA? How does the treatment of reasoning and proof vary across books for different courses and by different textbook series? Both the narrative and exercises in 20 textbooks dealing with the topics of exponents, logarithms, and polynomials were examined. Overall, attention to proof-related reasoning is greater in the narratives than in the exercises. The amount of proof-related reasoning in both the narratives and exercises varies by topic and textbook. Less than 6% of the exercises contain proof-related reasoning, with developing an argument as the most frequently occurring type of proof-related reasoning.

Background and Research Questions

Many mathematicians and mathematics educators consider reasoning and proof to be the heart of mathematics (Epp, 1998; Hanna, 2000; Herbst & Brach, 2006). As Hanna points out, "Students cannot be said to have learned mathematics, or even *about* mathematics, unless they have learned what a proof is" (p. 24). Despite this commitment to the teaching of proof, a considerable body of research has demonstrated that secondary school students, university students, and preservice teachers across the world have difficulty writing proofs in many content domains (Balacheff, 1988; Healy & Hoyles, 2000; Knuth, 2002; Recio & Godino, 2001; Seldon & Seldon, 2003; Senk, 1985; Thompson, 1996).

Cross-national studies of students' achievement (e.g., Burstein, 1993; McKnight et al., 1987; Valverde, Bianchi, Wolfe, Schmidt, Houang, 2002) have shown that the curriculum is a critical variable in accounting for differences in students' mathematics achievement, and textbooks are an important determinant of what teachers teach. If teachers want students to become fluent in their ability to engage in deductive reasoning or writing proofs, the curriculum should provide opportunities to study these concepts.

This study investigates opportunities to learn reasoning and proof provided to secondary school students by textbooks in the USA. In the early 20th century, deductive reasoning was only studied in geometry courses, and students memorized proofs similar to those in Euclids' *Elements* (Herbst, 2002). Recently, professional groups and national commissions have suggested that reasoning and proof are important at all grades and in all content domains (Kilpatrick, Swafford & Findell, 2001; National Council of Teachers of Mathematics [NCTM], 2000). In order to understand the extent to which recent recommendations have been implemented, this research examines reasoning and proof in textbooks for courses that *do not* focus on geometry, such as those focusing on topics from algebra or functions.

The specific research questions examined are: What is the nature and extent of reasoning and proof embodied in contemporary high school mathematics textbooks in the USA? How does the treatment of reasoning and proof vary across books for different courses and by different textbook series?

Development of the Research Framework

Across disciplines one finds many types of reasoning. Even within mathematics education one encounters spatial reasoning, statistical reasoning, inductive reasoning, and deductive reasoning. The framework for this study utilizes a construct we call *proof-related reasonin*g, which we base on the *Principles and Standards for School Mathematics* (NCTM, 2000) as well as on the work of other researchers. The Reasoning and Proof Standard recommends that instructional programs provide opportunities to do the following:

- Recognize reasoning and proof as fundamental aspects of mathematics;
- Make and investigate conjectures;
- Develop and evaluate mathematical arguments and proofs;
- Select and use various types of reasoning and methods of proof. (p. 56)

Secondary school textbooks in the USA usually consist of both passages to read, and problems to be solved. Initial pilot work examined instances of proof-related reasoning in several textbooks. It soon became apparent that we could recognize, with high reliability across coders, the last three bulleted points in both the narrative and exercises of textbook lessons. However, the first bullet point requires high inference, so researchers could not code it reliably. Examination of several sets of exercises suggested that noting other aspects of proof-related reasoning not identified in *Principles and Standards*, such as the use of counterexamples or different proof formats, might also be worth examining.

Many researchers distinguish levels of proof among students' work. For instance, Balacheff (1988) identifies three levels of proof (pragmatic proofs, intellectual proofs, demonstrations); Sowder and Harel (1998) identify three types of proof schemes (externally based, empirical, and analytic); and Miyazaki identifies six levels of proof. We wanted to build upon their analyses of students' work, in order to distinguish various types of reasoning and proof in U.S. textbooks. The conjectures and proofs we encountered in our examination of narratives and exercises in our pilot work fell into two broad categories. One type involves a specific case or cases, with justifications similar to what Balacheff calls *pragmatic*, Sowder and Harel call *empirical*, and Miyazaki calls *inductive reasoning*. A second type involves conjectures about general cases or general arguments or proofs similar to those Balacheff (1988) and Miyazaki (2000) call *demonstrations*, and Sowder and Harel (1998) call *analytic proofs*. Thus, we also developed codes to distinguish between general and specific types of proof-related reasoning.

Sample

The United States does not have a national curriculum or central control over textbook publishing. For this study we examined proof-related reasoning in 20 textbooks from six textbook series that are marketed nationwide. Specifically, we examined proof-related reasoning in Algebra 1, Algebra 2 and Precalculus textbooks published by three large companies (Glencoe, Holt, and Prentice-Hall), and one smaller company (Key). In addition, we examined eight textbooks produced by two different curriculum development projects: Courses 1 – 4 from the Core Plus Mathematics Project and books covering Algebra 1, Algebra 2, and two Precalculus courses called Functions, Statistics and Trigonometry (FST), and Precalculus and Discrete Mathematics (PDM) from the University of Chicago School Mathematics Project (UCSMP). Most of the 20 books are intended for students in Grades 9 – 12, although some Algebra 1 books are used in Grades 7 or 8. All books were published between 1996 and 2007. They include some of the most widely used textbooks in the U.S., and among them are some considered traditional and some considered to be standards-based (Senk & Thompson, 2003).

Methods

In each of the 20 textbooks we identified and examined all sections dealing with three broad topics where pilot work suggested that we might find some aspects of proof-related reasoning: (1) exponents, (2) logarithms, and (3) polynomial expressions, equations, and functions. For each of the focal topics, we created a list of properties that we expected would appear somewhere in each textbook series, and would likely need some justification, for example, the Product Property of Exponents or the Factor Theorem for Polynomials. Then we determined whether each of these properties is justified by a general argument, an argument based on a specific instance, left to the student, or not at all. We also noted whether there were opportunities for students to make conjectures related to these key properties. In each section that addressed one of these properties, we analyzed all exercises at the end of that lesson, counted the number that involve proof-related reasoning, and categorized the nature of the response expected. The codes used for proof-related exercises follow, with the second letter indicating whether the item was about a general (G) or specific (S) case.

- Develop an argument or proof (DG, DS)
- Evaluate an argument or proof (EG, ES)
- Investigate a conjecture (IG, IS)
- Make a conjecture (MG, MS)

In addition we also identified several other types of proof-related reasoning: finding a counterexample, completing a fill-in-the-blank argument, and correcting a (logical) mistake in an argument or proof.

We analyzed 151 sections containing 9792 exercises. Of these, 33 sections containing 2202 exercises dealt with exponents, 28 sections containing 1995 exercises dealt with logarithms, and 90 sections containing 5595 exercises dealt with polynomials. Figures 1 and 2 provide samples of coding for the narrative and exercises, respectively.

In order to justify the Power of a Power Property of Exponents, the Holt Algebra 1 book states the following.

$$(a^2)^3 = a^2 \bullet a^2 \bullet a^2 =$$
$$(a \bullet a) \bullet (a \bullet a) \bullet (a \bullet a) = a^6$$

No other justification is given. This justification is coded as an *argument based on a specific case*, because it applies only to the specific exponents 2 and 3.

In contrast, the following justification from the UCSMP Algebra 1 book developing a method for multiplying binomials is coded as a *general argument*, because it applies to all real numbers *a*, *b*, *c*, and *d*.

> Another way to show the pattern is to think of $(c+d)$ as a chunk and to distribute it over $(a+b)$ as follows:
>
> $(a+b)\bullet(c+d) = \boldsymbol{a}(c+d) + \boldsymbol{b}(c+d)$
>
> Now apply the Distributive Property twice more.
>
> $= ac + ad + bc + bd$.

Figure 1. Proof-Related Reasoning in the Narrative

Write a convincing argument to show why $3^0 = 1$ using the following pattern. $3^5 = 243$, $3^4 = 81$, $3^3 = 27$, $3^2 = 9$, …. (Glencoe Algebra 1, DS)

Prove the quotient rule for logarithms. (Prentice Hall Precalculus, DG)

State whether $3^5 \bullet 4^2 = 12^7$ is true or false. If it is false, explain why. (Key Algebra 2, IS)

Can the graph of a cubic polynomial function have two *x*-intercepts? If so, give an equation for such a function. If not, explain why not. (UCSMP Algebra 2, IG)

Figure 2. Proof-Related Reasoning in the Exercises

Results

Reasoning and Proof in the Narrative

The narrative portions of a textbook provide opportunities for students to read sample arguments or proofs during a lesson. Table 1 gives the average number of properties studied, and nature and extent of the justifications provided by course for the three focal topics.

Table 1. Average number of properties covered & nature of justifications by course and topic

		% of Properties Justified by Nature of Justification			
	Average no. of properties covered	General Argument	Specific Case	Left to Student	None
Algebra 1					
exponents	6.0	4.8	78.6	2.4	9.5
logarithms	0				
polynomials	5.4	63.2	10.5	0	26.3
Algebra 2					
exponents	5.8	11.4	11.4	2.9	74.3
logarithms	7.0	47.6	16.7	21.4	14.3
polynomials	12.8	20.8	19.5	7.8	49.4
Precalculus					
exponents	3.3	0	0	0	100
logarithms	9.3	44.6	7.7	20.0	27.7
polynomials	11.4	18.8	20.0	12.5	48.8
Overall					
exponents	5.0	6.0	37.0	2.0	53.0
logarithms	5.4	45.8	11.2	20.6	22.4
polynomials	9.8	28.2	17.9	8.2	44.6

Note: The number of properties examined by topic is: exponents, 7; logarithms, 16; and polynomials, 21. Seven books are classified as Algebra 1, 6 as Algebra 2, and 7 as Precalculus. Some justifications were classified as "other," so percentages may not add to 100%.

On average, Algebra 1 books address more properties of exponents than do Algebra 2 or Precalculus books. When properties of exponents appear in the later textbooks, they tend to be presented as review. Typically, properties of exponents are justified with specific instances in Algebra 1 textbooks and are not justified at all in most Algebra 2 and Precalculus textbooks. Few properties of exponents are left to the student to justify in any of the textbooks. Across all courses and books, only 6% of exponent properties are justified with a general argument, so most students using the textbooks we examined never have an opportunity to read general proofs of the properties of exponents.

No Algebra 1 books we examined addressed logarithms. Almost half of the properties in the lessons about logarithms in Algebra 2 books provide general arguments for these properties, and a slightly lower portion are proved in the narratives of Precalculus books. About 21% of the logarithm properties are left to the student to be proved in the exercises. Across all courses and books, almost half of logarithm properties are proved with a general argument.

On average, Algebra 1 books contain fewer properties of polynomials than do books for Algebra 2 or Precalculus. However, a much higher percentage of polynomial properties are justified with general arguments in Algebra 1 than in Algebra 2 or Precalculus. Although more than 60% of the polynomial properties in Algebra 1 books are justified with a general argument, about 50% of polynomial properties in Algebra 2 and Precalculus books are not justified in at all. This may be due to the fact that some polynomial theorems that are justified in Algebra 1, e.g. the Theorem for Squaring a Binomial or the Quadratic Formula Theorem are stated without proof in Algebra 2 and Precalculus books. In other cases, proofs of theorems about polynomials included in Algebra 2 and Precalculus textbooks, such as the Fundamental Theorem of Algebra, may require mathematical knowledge beyond the scope of the course.

The percentage of properties treated in the narrative and the nature of the justifications varies considerably by publisher and individual textbook. Space does not permit an examination across all three topics. Table 2 shows variation in the nature of the treatment of the 21 polynomial properties in the narratives of the 18 textbooks that included this topic.

Table 2. Number of polynomial properties covered and nature of justification by course

		% of Properties Covered by Nature of Justification			
	No. of properties covered	General Argument	Specific Case	Left to Student	None
Algebra 1					
Glencoe	10	70.0	20.0	0.0	10.0
Holt	9	33.3	11.1	0.0	55.6
Key	2	50.0	0.0	0.0	50.0
Prentice Hall	9	77.8	0.0	0.0	22.2
UCSMP	8	75.0	12.5	0.0	12.5
Algebra 2					
Core Plus 3	7	14.3	0.0	14.3	71.4
Glencoe	16	18.8	12.5	0.0	56.3
Holt	16	12.5	50.0	0.0	37.5
Key	8	37.5	25.0	12.5	25.0
Prentice Hall	14	14.3	14.3	0.0	71.4
UCSMP	16	31.3	6.3	25.0	37.5
Precalculus					
Core Plus 4	6	0.0	16.7	66.7	16.7
Glencoe	13	0.0	23.1	0.0	76.9
Holt	16	18.8	25.0	6.3	50.0
Key	6	16.7	83.3	0.0	0.0
Prentice Hall	21	9.5	9.5	9.5	71.4
UCSMP FST	11	54.5	9.1	9.1	27.3
UCSMP PDM	7	42.9	0.0	28.6	28.6

All books provide some justifications for some of the polynomial properties covered. Among Algebra 1 books, when properties are covered, four of the five books provide a general argument for at least 50% of the properties. However, at the Algebra 2 and Precalculus levels, with one exception (UCSMP FST), general arguments are provided for less than 50% of the properties covered. Neither the Glencoe Precalculus nor the Core Plus 4 provide and general arguments, but whereas the Core Plus 4 leaves two thirds of the properties it covers to the student, the Glencoe Precalculus provides no justification for over three fourths of the covered properties. The results in Table 2 also illustrate the lack of uniformity of content covered among U.S. textbooks, with between 2 and 8 of the polynomial properties covered in Algebra 1, between 7 and 16 at Algebra 2, and between 6 and 21 at Precalculus.

Reasoning and Proof in Exercises

The exercises in a textbook provide evidence of the opportunities students have to engage in proof-related reasoning themselves, e.g., to make or investigate conjectures, or to develop or evaluate mathematical arguments. Table 3 summarizes the type of proof-related reasoning identified in the exercises from lessons dealing with the three focal topics.

Table 3. Nature of proof-related reasoning in exercises by course, textbook series, and overall

				Percent of exercises by type of proof-related reasoning				
	No. of lessons coded	No. of exercises analyzed	Percent of exercises requiring proof-related reasoning	Develop Argument	Evaluate Argument	Investigate Conjecture	Make Conjecture	Other[a]
Course								
Algebra 1	43	2844	3.3	1.4	0.0	0.9	0.3	0.7
Algebra 2	58	3937	5.2	2.1	0.0	1.7	0.5	0.9
Precalculus	50	3011	7.8	4.7	0.2	1.4	0.8	0.7
Textbook Series								
Core Plus	9	532	14.1	6.0	0.6	0.2	4.0	3.4
Glencoe	29	2117	3.7	1.8	0.0	1.0	0.1	0.8
Holt	25	2042	3.7	2.3	0.0	1.1	0.1	0.2
Key	17	916	8.0	3.4	0.1	3.3	1.0	0.2
Prentice Hall	29	2446	5.6	2.7	0.0	1.9	0.1	0.9
UCSMP	42	1739	6.2	2.9	0.3	0.9	0.9	1.3
Overall	151	9792	5.6	2.7	0.1	1.4	0.5	0.9

[a] Other proof-related reasoning includes finding a counterexample, correcting a mistake, filling-in-the-blanks of an incomplete proof, and exercises that are double coded.

The number of proof-related exercises is quite low, averaging less than 6%, and variation occurs by course. For instance, the percent of exercises that are proof-related ranges from approximately 3% in Algebra 1 books to almost 8% in Precalculus books. The percent of exercises requiring students to develop an argument increases with the grade level for which the books are intended, i.e., from about 3% in Algebra 1 to 5% in Algebra 2 to 8% in Precalculus.

Variation also occurs across textbook series. The percent of proof-related exercises ranges from approximately 4% in the Glencoe and Holt series to approximately 14% in the Core Plus series. The Core Plus books have the highest percent of exercises requiring students to develop an argument and to make a conjecture; the books published by Key have the largest percent of exercises requiring students to investigate a conjecture, that is, to decide if a statement is always true. Only three textbook series (Core Plus, Key, UCSMP) have any percent of exercises requiring students to investigate a conjecture.

Figures 3 and 4 disaggregate data on proof-related reasoning in Table 3 for individual textbooks. Eighteen textbooks contain some sections about polynomials. Because the percent for the nature of proof-related reasoning in some books is quite low, we collapsed the data for Develop and Evaluate an Argument into a single category, and those for Make and Investigate a Conjecture into another single category. Figure 3 shows the percent of exercises about polynomials in sections in these books for various types of proof-related reasoning.

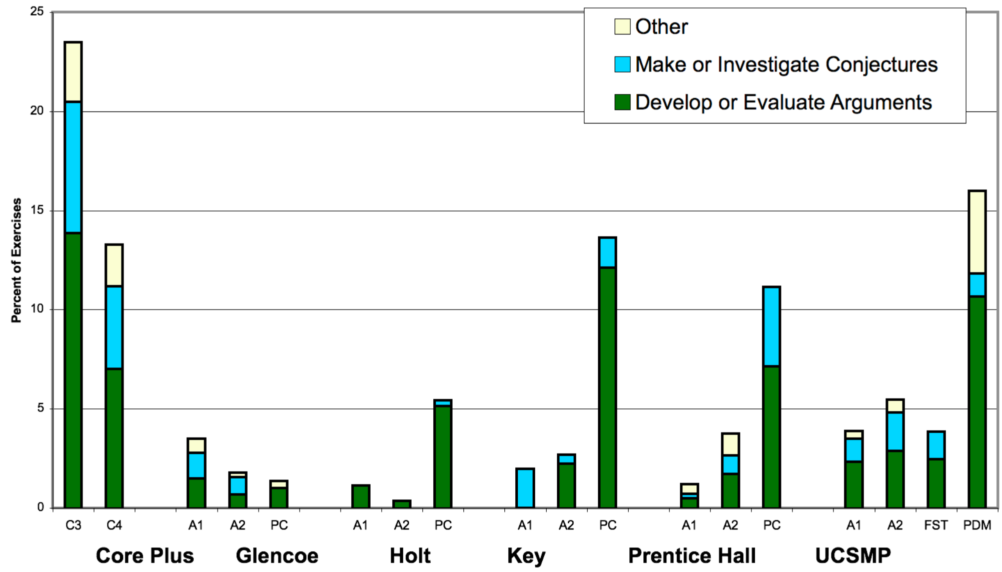

Figure 3. Percent of Exercises in Sections on Polynomials by Nature of Reasoning

Five books devote at least 10% of their exercises on polynomials to proof-related reasoning, with the Course 3 book from Core Plus being the highest at nearly 24%. Two books devote approximately 5% of their exercises to proof-related reasoning. In the other 11 books less than 5% of the exercises about polynomials require proof-related reasoning, with the Algebra 2 book published by Holt having the lowest percentage (less than 0.4%). In general, across all books more attention is paid to developing or evaluating arguments than to making or investigating conjectures. However, there is one exception – the Algebra 1 book by Key: less than 2% of the exercises about polynomials in this book contain proof-related reasoning, but virtually all of them involve making or investigating a conjecture.

Figure 4 displays variation in the type of exercises in sections about polynomials in another way – whether the proof-related reasoning involves a general or specific case. In three books the percentage of exercises devoted to general arguments or conjectures exceeds 10% – Courses 3 and 4 from Core Plus and the Precalculus book from Key. In the other 15 books no more than 5% of the exercises on polynomials involve making a general argument or a conjecture about a general case.

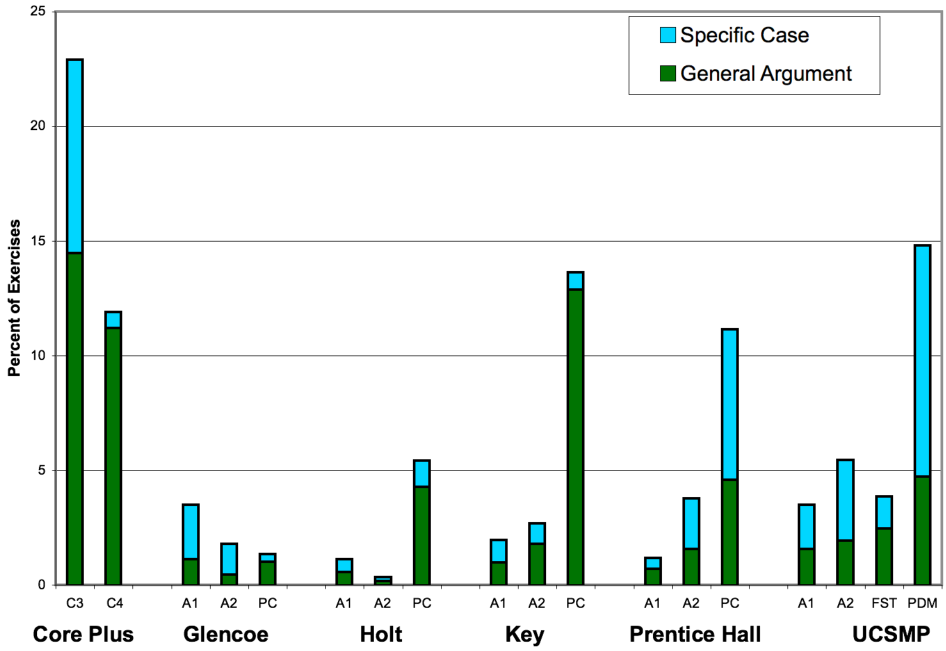

Figure 4. Percent of Exercises in Sections on Polynomials by Specific or General Argument

Summary, Conclusions, and Discussion

Because the U.S. has no national mathematics curriculum, the intended curriculum is embodied in textbooks. This research investigated the extent and nature of proof-related reasoning in the narrative and exercises of 151 lessons in 20 high school textbooks in the USA. In the narrative of the sections examined, properties of polynomials were more likely than properties of exponents and less likely than properties of logarithms to be justified with a general argument. Overall, the narratives contain more proof-related reasoning than do the exercise sets. In other words, assuming that students read the narratives, they read far more justifications than they are asked to develop. However, only studies of how these textbooks are used in classrooms, will allow researchers to examine the implemented curriculum.

All textbook series provide some opportunity to develop arguments, but only about 3% of exercises examined are of this type. Exercises requiring students to investigate or make conjectures occur even less frequently; and virtually no exercises ask students to evaluate arguments. Because there is little, if any, research linking the type or amount of proof-related reasoning in textbooks to students' ability to reason mathematically, it is difficult to conclude whether the nature and extent of proof in the textbooks we examined is sufficient. However, based on our data it seems that the Reasoning and Proof Standard (NCTM, 2000) has not been widely implemented in high school textbooks. Because many research studies have shown that writing proofs is difficult for many students, we recommend that exercise sets feature a greater variety and higher percentage of problems that require proof-related reasoning.

In some textbooks the exercises ask for arguments about specific cases more frequently than for general arguments. This may lead some students to believe that giving an argument for a specific case of a property is sufficient to prove the property. Because research has shown that many students confuse the relations between examples, arguments based on specific cases, and general arguments, textbook authors and teachers need to pay more attention to this issue.

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