'COGNITIVE CONFLICT' AS A MECHANISM FOR SUPPORTING DEVELOPMENTAL PROGRESSIONS IN STUDENTS' KNOWLEDGE ABOUT PROOF[1]

Andreas J. Stylianides
University of Oxford, U.K.
andreas.stylianides@education.ox.ac.uk

Gabriel J. Stylianides
University of Pittsburgh, U.S.A.
gstylian@pitt.edu

ABSTRACT. A common challenge faced by the 'cognitive conflict' approach to mathematics teaching is that students often possess 'contradictory understandings' (from a mathematical point of view) without feeling the 'intellectual need' to address the inconsistencies in their understandings. In other words, when instruction engages students in mathematical situations where some of their existing understandings about an idea or a topic no longer hold, students tend to treat the contradictions as exceptions. In this article we present a theoretical framework that aimed to cast light on conditions that increase the likelihood of students to experience a cognitive conflict as a result of a contradiction engineered by instruction. Also, we use a research-based instructional sequence to exemplify the application of this framework in designing instructional sequences that aim to use cognitive conflict as a mechanism for supporting developmental progressions in students' knowledge about proof.

In this article we are concerned with the use of *cognitive conflict* (e.g., Hadas et al., 2000; Tirosh & Graeber, 1990; Zaslavsky, 2005) as a mechanism for supporting an 'intellectual need' (Harel, 1998) for developmental progressions in students' mathematical knowledge in general and knowledge about proof in particular. Zaslavsky (2005, pp. 299-300) provided a detailed discussion about the roots of the notion of cognitive conflict in Dewey's concept of reflective thinking and its relations to psychological theories such as Piaget's equilibration theory, Festinger's theory of cognitive dissonance, and Berlyne's theory of conceptual conflict. A major goal of the 'conflict teaching' approach to mathematics teaching is to help students reflect on their current mathematical understandings, confront contradictions that arise in situations where some of these understandings no longer hold, and recognize the importance (need) of modifying these understandings to resolve the contradictions.

THEORETICAL FRAMEWORK

Potential Conflict versus Cognitive Conflict: The Notion of 'Pivotal Counterexamples'

A common challenge faced by the cognitive conflict approach to mathematics teaching is that students often possess 'contradictory understandings' (from a mathematical point of view) without feeling the intellectual need to address the inconsistencies in their understandings (Zazkis & Chernoff, in press). In other words, when instruction engages students in mathematical situations where some of their existing understandings about an idea or a topic no longer hold, students often do not see the importance (or necessity) to engage in a process of modifying their understandings to resolve the contradictions and they tend to treat the contradictions as exceptions. Such mathematical situations present an opportunity for a *potential conflict*, which may or may not

[1] The two authors had an equal contribution to the preparation of this article. The work reported in the article was supported by funds from the Spencer Foundation to both authors (Grant Numbers: 200700100 and 200800104) and the UK's Economic and Social Research Council to the first author (Grant Number: RES-000-22-2536). The opinions expressed in the article are those of the authors and do not necessarily reflect the position, policy, or endorsement of the aforementioned funding agencies.

develop to a *cognitive conflict* for students (ibid). In what ways can instruction transform a potential conflict that is engineered by instruction into a cognitive conflict for the students who engage with the tasks that comprise an instructional sequence?[2]

A way to address this issue is to strategically incorporate into the instructional sequences collections of *pivotal counterexamples* (Zazkis & Chernoff, in press). According to Zazkis and Chernoff, a counterexample is pivotal for a student if it creates a turning point in the student's cognitive perception, that is, if it creates a dissonance in the student's incorrect or incomplete understandings of a particular topic or idea or, in other words, if it helps develop a potential conflict into a cognitive conflict for the student. Although a counterexample is a mathematical concept, a pivotal counterexample is a pedagogical concept (ibid). Therefore, unlike a counterexample that can be determined universally, a pivotal counterexample can only be anticipated and recognized as such only after the implementation of the instructional sequence to which it belongs (ibid). The distinction between the mathematical notion of counterexample and the pedagogical notion of pivotal counterexample offers a useful theoretical tool to explain why some counterexamples that are presented to students with the intent to create cognitive conflict are dismissed by students and are treated as exceptions. Yet, the distinction itself does not cast light on the conditions under which a counterexample has good potential to become pivotal for students. Understanding these conditions has implications for the design of instructional sequences which aim to use pivotal counterexamples to create cognitive conflicts that can support particular developmental progressions in students' knowledge. Next we discuss two such conditions.

Conditions that Increase the Likelihood of a Counterexample to become Pivotal for Students

Condition #1: The Relation between Pivotal Counterexamples and Example Spaces

The first condition is derived from the work Zazkis and Chernoff (in press) and concerns Watson and Mason's (2005) notion of *example spaces*, that is, collections of examples that fulfill a particular function. According to Zazkis and Chernoff, a counterexample that becomes pivotal for a student often falls (1) outside the immediately available and easily accessible example space of the student at a given time, i.e., outside the student's *personal example space* (Watson & Mason, 2005, p. 76), and (2) inside an example space that is within the student's ability to understand the given time (perhaps after some scaffolding from the instructor), i.e., inside the student's *personal potential example space* (Watson & Mason, 2005, p. 76). A pivotal counterexample can expand the boundaries of a student's personal example space to include a space that was previously within the student's personal potential example space; this expansion can push, in turn, the boundaries of the student's personal potential example space (Zazkis & Chernoff, in press). Zazkis and Chernoff suggested further that a reason for which a counterexample can fail to become pivotal for a student is that it falls outside the student's personal potential example space at a given time, even though it may belong to a *conventional example space*, that is, an example space "as generally understood by mathematicians and as displayed in textbooks, into which the teacher hopes to induct his or her students" (Watson & Mason, 2005, p. 76).

The theoretical ideas discussed above are not particular to the teaching of proof due to the generic meaning of example spaces. To make these ideas specific to proof, we propose a connection between the notion of example spaces and Harel and Sowder's (1998) notion of *justification schemes*, which signifies what argument convinces a student and what argument the

[2] By 'instructional sequence' we mean a series of tasks and associated instructor actions in implementing the tasks.

student offers to convince others (i.e., what counts as a proof from the student's point of view). The connection between the two notions gives rise to what we call *example spaces for validation.* The main function of collections of examples in this special kind of example spaces is that of validating a mathematical generalization. In the context of a mathematical generalization that is to be validated by a student, the student's justification scheme reflects his or her view on the collection of examples that are considered sufficient for the validation of the generalization, i.e., it reflects the student's personal example space for validation. The link between justification schemes and example spaces for validation allows the use of existing theory on pivotal counterexamples (as described above) to inform the design of instructional sequences that would aim to help students progress towards more advanced justification schemes. Specifically, it allows the design of instructional sequences that would use pivotal counterexamples in order to achieve (by means of cognitive conflict) expansions in students' personal example spaces for validation, thereby supporting parallel developmental progressions in students' justification schemes.

In Table 1, we present (in increasing levels of mathematical sophistication) and define major kinds of justification schemes and corresponding example spaces for validation. The first two kinds of justification schemes in the table – *naïve empirical justification scheme* and *crucial experiment justification scheme* – are a partitioning of Harel and Sowder's (1998) 'empirical justification scheme' and derive from two special kinds of empirical arguments that Balacheff (1988) called 'naïve empiricism' and 'crucial experiment.' The corresponding example spaces for validation for these justification schemes are the *naïve empirical example space for validation* and the *crucial experiment example space for validation*. We note also that the *conventional justification scheme*, which is consistent with a mathematical notion of what constitutes a proof, can be viewed as the most advanced version of the *non-empirical justification scheme* in the table. A student who possesses the non-empirical justification scheme recognizes that the validation of a mathematical generalization has to consider all elements of the domain of discourse of the generalization, i.e., the student's personal example space for validation is the *conventional example space for validation*. Thus the student rejects empirical arguments as secure methods for validating a mathematical generalization, but this does not mean necessarily that the student recognizes also that proof can fulfill this function (this would be the case if the student possessed the conventional justification scheme). A student with the non-empirical justification scheme may consider, for example, that mathematical generalizations over infinite sets cannot be validated, because it is impossible to check one-by-one all possible cases in their domain of discourse.

Condition #2: The Relation between Pivotal Counterexamples and Conceptual Awareness Pillars

The second condition to increase the likelihood of a counterexample in the domain of proof to become pivotal for students concerns the creation of instructional situations that would help students increase their 'awareness' (Mason, 1998)[3] of the nature of their personal example spaces for validation (or corresponding justification schemes) both prior to and following the implementation of a counterexample. In order to create instructional situations that would increase students' awareness (both explicit and potential) of their personal example spaces for validation (or corresponding justification schemes), we propose the strategic incorporation into the instructional sequences collections of what we call *conceptual awareness pillars*. This is a theoretical construct that we introduce in this article and we define broadly to describe instructional activities that aim to

[3] According to Mason (1998), "[b]eing aware is a state in which attention is directed to whatever it is that one is aware of" (p. 254). Mason noted further that "this idiom is used to refer both to being explicitly aware and to being potentially aware" (ibid).

direct students' 'attention' (cf. Mason, 1998) to their understandings or conceptions of a particular mathematical topic or idea (students' personal examples spaces being a case in point).

Conceptual awareness pillars can take different forms at both the individual and social levels. An example of a conceptual awareness pillar at the individual level is to ask students to reflect independently on a specific prompt that relates to a particular topic or idea such as the following: "Is there a specific number of examples that you need to check in order to validate a mathematical generalization?" This prompt could be used to direct students' attention to their conceptions of what counts as sufficient evidence for the validity of a generalization in mathematics, thereby making them more aware of their justification schemes, even though they may not be explicitly aware of the existence of such schemes. An example of a conceptual awareness pillar at the social level is the process of asking students to reflect as a group on a specific prompt as explained earlier. Another example at the social level is the initiation by the instructor of a 'situation for institutionalization' (Brousseau, 1981), that is, a situation that aims to point out and give an official status to pieces of knowledge that had been constructed during an activity in the class and that could be used in future work (see Balacheff, 1990, p. 260). In the context of our work, these pieces of knowledge could reflect the dominant understandings or conceptions of a class as a community of learners about a particular topic or idea and did not necessarily have to be consistent with conventional mathematical knowledge.

EXEMPLIFYING THE APPLICATION OF THE THEORETICAL FRAMEWORK IN THE DESIGN OF AN INSTRUCTIONAL SEQUENCE

Background

In this section we describe an instructional sequence whose design was informed by the theoretical framework described earlier. This instructional sequence is part of a larger instructional sequence that we developed and implemented (with slight variations) in two different design experiments which aimed to promote students' understanding of proof. The first design experiment was conducted in an undergraduate mathematics course for prospective elementary teachers in the United States and the second is conducted currently in two secondary mathematics classrooms in England (completion date: July 2008). The full version of the instructional sequence as implemented in the first design experiment was reported in Stylianides and Stylianides (2008).

In both design experiments, the part of the instructional sequence on which we focus in this article (hereafter referred to as 'focal instructional sequence') was implemented at the beginning of the classroom communities' work on proof. Pre-course measures in both experiments indicated that most students in the classroom communities possessed empirical justification schemes, primarily naïve empirical in nature but also of the form of crucial experiment (cf. Table 1). Therefore, the primary goal of the focal instructional sequence was to help the students begin to realize the limitations of empirical arguments as methods for validating mathematical generalizations, i.e., to help them move towards the non-empirical justification scheme (cf. Table 1). In accordance with the theoretical framework presented earlier, the focal instructional sequence aimed to use pivotal counterexamples to achieve (by means of cognitive conflict) expansions in students' personal example spaces for validation, thereby supporting parallel developmental progressions in students' justification schemes.

Figure 1 offers an overview of key aspects of the focal instructional sequence, including the names of the two tasks that comprised the sequence (each of which incorporated a potential pivotal counterexample), the conceptual awareness pillars that we used in order to increase the likelihood that each potential conflict would develop to cognitive conflict for the students, and the developmental progressions in students' justification schemes that each cognitive conflict was intended to support. Table 2 complements figure 1 by summarizing major features of the two tasks that comprised the focal instructional sequence and corresponding major elements of the 'hypothetical learning trajectory' (Simon, 1995)[4] of a classroom community in which the instructional sequence was implemented. Before we elaborate on key aspects of Table 2, we clarify three issues related to our focus in this article on *a classroom community's learning trajectory*.

The first issue concerns the members of the classroom community. We considered the classroom community to consist primarily of the students. The instructor had a special membership status in this community as the representative of the discipline of mathematics and as the person who had a special role to play in trying to connect students with broader mathematical knowledge (see, e.g., Lampert, 1992; Yackel & Cobb, 1996).

The second issue concerns the relationship between individual learners' developmental progressions and advancements in the 'taken-as-shared' (see, e.g., Cobb & Yackel, 1996) knowledge within a classroom community. Lampert (1992) suggested that we think about this relationship in terms of a continuum, proposing that we "consider every act of 'knowing' as occurring somewhere on this continuum" (p. 310). The knowledge that can be considered as shared within the community does not necessarily reflect the understanding of each learner, for the individual learners who make up the classroom community "go their separate ways with whatever knowledge they have acquired" (ibid). Accordingly, when we talk about a classroom community's hypothetical learning trajectory, we do not imply that each learner in the community was anticipated to follow the same learning trajectory. Rather, we call attention to the advancements in the 'taken-as-shared' knowledge that we expected to see reflected in the community's public/collective work. We clarify further that our focus on the classroom community level does not suggest that we consider the individual level to be less important. We agree with Lampert (1992) that "[t]eaching and learning need to take account both of what is accomplished by individuals and what is understood to be 'true' within the classroom discourse" (p. 310). Along similar lines, one can say that research on teaching and learning needs to take into account both what is accomplished by individuals and what is understood to be shared within the classroom discourse. Therefore, our characterizations of the classroom community's 'actual learning trajectory' (Leikin & Dinur, 2003)[5] were informed not only by what happened in whole groups discussions, but also by data on small group and individual student constructions.

Finally, our analysis of the implementation of the focal instructional sequence in classroom communities that participated in the two design experiments we described earlier showed that faithful implementation of the instructional sequence yielded close matching between the communities' hypothetical and actual learning trajectories. Next we report the learning trajectory followed by classroom communities where such faithful implementation of the instructional sequence took place. Space limitations do not allow us to offer empirical evidence to support our claim of close matching between actual and hypothetical learning trajectories. An expanded version of this article (Stylianides & Stylianides, in review), which is available upon request, offers

[4] The term 'hypothetical learning trajectory' refers to the learning routes that students were anticipated to follow towards the achievement of specific learning goals, as a result of the implementation of the instructional sequence in a classroom community.

[5] The term 'actual learning trajectory' refers to the learning routes that students seemed to have actually followed as a result of the implementation of the instructional sequences in a classroom community.

detailed presentation and analysis of the implementation of the focal instructional sequence in our design experiment at the undergraduate level.

Design Features of the Focal Instructional Sequence that aimed to provoke two Cognitive Conflicts in order to Support Developmental Progressions in Students' Justification Schemes

In the expanded version of the instructional sequence, there was a task prior to the first task in figure 1 (task #0) that aimed to provoke students' propensity to validate mathematical generalizations on the basis of naïve empiricism (see Stylianides & Stylianides [2008] for details). Before implementing the first task in the focal instructional sequence, the instructor implemented conceptual awareness pillars #1 and #2 as shown in figure 1. CAP #1 asked the students to reflect independently on the method they used to validate the generalization in task #0 (primarily naïve empirical). CAP #2 was a situation for institutionalization in which the instructor reported to the students the dominant (common) validation method based on analysis of their individual reflections in CAP #1. These two CAPs helped the students become more aware of their conceptions about methods for validating mathematical generalizations, thereby making it more likely that they would experience a cognitive conflict when their validation methods would prove problematic in the first task of the focal sequence (cf. condition #2 in the theoretical framework).

Next we describe separately how each task in the focal instructional sequence provoked a cognitive conflict to support developmental progressions in students' justification schemes. We recommend that the reader reads the text in parallel to relevant parts of figure 1 and Table 2.

The Circle and Spots Problem

The Circle and Spots Problem (figure 2) challenged students' reliance on naïve empirical arguments by: (1) engaging them in the identification of a numerical pattern ($\alpha_n = 2^{n-1}$) that they trusted on the basis of naïve empiricism and applied in a case (n=15) that was practically difficult for them to check empirically, and (2) creating a cognitive conflict for them with the discovery of a counterexample to the pattern for n=6. This counterexample fell immediately outside the naïve empirical example space for validation, which (based on our experience with implementing the Circles and Spots Problem in previous years) included typically the first four or five terms of the sequence. Thus the counterexample was pivotal for many students, as it fell outside their personal example space for validation the given time, but within their potential example space for validation (cf. condition #1 in the theoretical framework). The first two conceptual awareness pillars that preceded students' engagement with the Circles and Spots Problem helped make the counterexample pivotal for students (cf. condition #2 in the theoretical framework).

Many students resolved the cognitive conflict by considering a more sophisticated form of empirical argument than naïve empiricism (crucial experiment) when trying to validate a mathematical generalization. Since the counterexample in the Circle and Spots Problem corresponded to a term of the associated number sequence that students could practically check, many students progressed at this point to the crucial experiment example space for validation, thinking that their selection of cases to check when trying to validate a generalization had to become more strategic so that possible counterexamples would be revealed. The 'Monstrous Counterexample' Illustration (figure 5)[6] aimed to provoke a new developmental progression in

[6] The name 'Monstrous Counterexample' was not mentioned during the implementation of the focal instructional sequence. We use this name in the article for purposes of easy reference to the illustration.

students' conception of what it means to validate a mathematical generalization by designing a new potential conflict in students' activity.

The Monstrous Counterexample Illustration

The Monstrous Counterexample Illustration presented a pattern that held for a huge number of cases (of the order of septillions) but ultimately failed. This was a pivotal counterexample for many students because it fell outside the crucial experiment example space for validation but within the conventional example space for validation, which, at this stage of the instructional sequence, was the new potential example space for validation for many students (cf. condition #1 in the theoretical framework). To make this counterexample become pivotal for more students (and thus for them to experience a new cognitive conflict), we inserted in the instructional sequence two additional conceptual awareness pillars (cf. condition #2 in the theoretical framework). These conceptual awareness pillars aimed to direct students' attention to their current conceptions of what it meant to validate a mathematical generalization (CAP #3 in figure 2) and to the implications of the Monstrous Counterexample Illustration for the veracity of these conceptions (CAP #4). Both conceptual awareness pillars took the form of group reflections on the following (fictional) pupil statement: *"Checking 5 cases is not enough to trust a pattern in a problem. Next time I work with a pattern problem, I'll check 20 cases to be sure."*

Many students resolved the cognitive conflict by progressing to the conventional example space for validation, recognizing that the validation of a mathematical generalization needs to consider all elements in the domain of discourse of the generalization. This recognition supported also recognition by many students of empirical arguments as insecure methods for validation (cf. non-empirical justification scheme in Table 1). The rejection of empirical arguments as secure methods for validation created, in turn, an intellectual need in many students to learn about conventional (secure) methods for validation. This need was expressed in different ways: a feeling of frustration that one could not validate mathematical generalizations over infinite sets, general distrust in mathematical generalizations, etc. At this point the instructor as the representative of the mathematical community in the classroom helped students address this intellectual need by introducing them to the notion of proof in mathematics: explaining to them the importance of proof as a secure method followed by mathematicians to validate mathematical generalizations.

CONCLUSION

In this article we contributed to theory about the use of cognitive conflict as a mechanism for supporting an 'intellectual need' (Harel, 1998) for developmental progressions in students' mathematical knowledge, which in the domain of proof can be conceptualized as progressions towards more sophisticated justification schemes. In particular, we proposed and exemplified a framework that aims to cast light on conditions that increase the likelihood of students to experience a cognitive conflict as a result of a contradiction (counterexample) engineered for this purpose by instruction. The framework discussed two such conditions that synthesized and extended prior research work, using the existing notions of pivotal counterexamples (Zazkis & Chernoff, in press) and example spaces (Watson & Mason, 2005) in relation to the notions of example spaces for validation and conceptual awareness pillars that we introduced in this article. The framework suggested that whether students experience a cognitive conflict as a result of their confrontation with a counterexample depends on the extent to which (1) the counterexample is in accord with students' example spaces (for validation) (Zazkis & Chernoff, in press), and (2) instruction supported students to become aware of their conceptions (by means of conceptual awareness pillars) both prior to and following the implementation of the counterexample.

REFERENCES

Balacheff, N. (1988). Aspects of proof in pupils' practice of school mathematics. In D. Pimm (Ed.), *Mathematics, teachers and children* (pp. 216-235). London: Hodder & Stoughton.

Balacheff, N. (1990). Towards a *problématique* for research on mathematics teaching. *Journal for Research in Mathematics Education, 21,* 258-272.

Brousseau, G. (1981). Problèmes de didactique des décimaux. *Recherches en Didactique des Mathématiques, 2,* 37-128.

Davis, P. J. (1981). Are there coincidences in mathematics? *American Mathematical Monthly, 88*, 311-320.

Hadas, N., Hershkowitz, R., & Schwarz, B. (2000). The role of contradiction and uncertainty in promoting the need to prove in dynamic geometry environments. *Educational Studies in Mathematics, 44*, 127-150.

Harel, G. (1998). Two dual assertions: The first on learning and the second on teaching (or vice versa). *American Mathematical Monthly, 105*, 497-507.

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. In A. H. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *Research in collegiate mathematics education III* (pp. 234–283). Providence, RI: American Mathematical Society.

Lampert, M. (1992). Practices and problems in teaching authentic mathematics. In F. K. Oser, A. Dick, & J. Patry (Eds.), *Effective and responsible teaching: The new synthesis* (pp. 295–314). San Francisco: Jossey-Bass Publishers.

Leikin, R., & Dinur, S. (2003). Patterns of flexibility: Teachers' behavior in mathematical discussion. *Proceedings of the 3rd Conference of the European Society for Research in Mathematics Education.* Bellania, Italy. Retrieved July 22, 2007, from http: //www.dm.unipi.it/~didattica/CERME3/proceedings/Groups/TG11/TG11_Leikin_cerme3.pdf

Mason, J. (1998). Enabling teachers to be real teachers: Necessary levels of awareness and structure of attention. *Journal of Mathematics Teacher Education, 1,* 243-267.

Mason, J., Burton, L., & Stacey, K. (1982). *Thinking mathematically.* London: Addison-Wesley.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*, 114-145.

Stylianides, G. J., & Stylianides, A. J. (2008). Enhancing undergraduate students' understanding of proof. *Article to appear in the Electronic Proceedings of the 11th Conference on Research in Undergraduate Mathematics Education*. San Diego, California.

Stylianides, G. J., & Stylianides, A. J. (in review). Enhancing students' understanding of proof.

Tirosh, D., & Graeber, A. O. (1990). Evoking cognitive conflict to explore preservice teachers' thinking about division. *Journal for Research in Mathematics Education, 21*(2), 98-108.

Watson, A., & Mason, J. (2005). *Mathematics as a constructive activity: Learners generating examples.* Mahwah, NJ: Lawrence Erlbaum.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education, 27*, 458–477.

Zaslavsky, O. (2005). Seizing the opportunity to create uncertainty in learning mathematics. *Educational Studies in Mathematics, 60*, 297-321.

Zazkis, R., & Chernoff, E. J. (in press). What makes a counterexample exemplary? *Educational Studies in Mathematics.*

Table 1
Justification Schemes and Corresponding Example Spaces for Validation in Increasing Levels of Mathematical Sophistication

Justification scheme	Example space for validation
Naïve empirical justification scheme: special kind of 'empirical justification scheme'[a] in which a mathematical generalization is validated on the basis of 'naïve empiricism'[b]	*Naïve empirical example space for validation:* few terms of the sequence which corresponds to the mathematical generalization, typically a collection of terms that are practically convenient to check
Crucial experiment justification scheme: special kind of 'empirical justification scheme'[a] in which a mathematical generalization is validated on the basis of 'crucial experiment'[c]	*Crucial experiment example space for validation:* few terms of the sequence which corresponds to the mathematical generalization, selected based on some kind of rationale (e.g., a strategy for revealing possible counterexamples)
Non-empirical justification scheme: students who possess this justification scheme recognize any kind of empirical argument as an insecure method for validating a mathematical generalization • *Conventional justification scheme:* o the most advanced version of the non-empirical justification scheme o students who possess this justification scheme recognize proof as a secure method for validating a mathematical generalization	*Conventional example space for validation:* all terms of the sequence which corresponds to the mathematical generalization

Notes: [a] Students who possess the 'empirical justification scheme' (Harel & Sowder, 1998) consider empirical arguments as proofs of mathematical generalizations, whereby empirical arguments we mean arguments that offer inconclusive evidence for the truth of a generalization by verifying its truth only for a proper subset of all the cases covered by the generalization.

[b] 'Naïve empiricism' (Balacheff, 1988) refers to the special kind of empirical argument that validates a mathematical generalization based on the confirming evidence offered by few cases, typically a collection of cases that are practically convenient to check.

[c] 'Crucial experiment' (Balacheff, 1988) refers to the special kind of empirical argument that validates a mathematical generalization based on the confirming evidence offered by few cases that are selected according to some kind of rationale (e.g., a strategy for revealing possible counterexamples).

Table 2
Major Features of the Circle and Spots Problem and the Monstrous Counterexample Illustration and Corresponding Elements of the Classroom Community's Hypothetical Learning Trajectory

Task	Major task features	Major elements of the classroom community's hypothetical learning trajectory
Circle and Spots Problem	• Includes first five cases that give rise to a numerical pattern which does not apply for later cases	• Students explore particular cases and identify a pattern (invalid)
	• Asks for the value of the corresponding number sequence for a large case that is practically difficult to check empirically	• Students validate the pattern on the basis of empirical arguments in the form of *naïve empiricism* (students' personal example space for validation is *naïve empirical*)
	• Includes *potential pivotal counterexample #1* o The counterexample falls outside the *naïve empirical example space for validation*, but is within the students' potential example space for validation	• Students experience *cognitive conflict #1.* Corresponding *developmental progression*: o Students consider also more advanced empirical arguments in the form of *crucial experiment* (students progress to the *crucial experiment example space for validation*)
Monstrous Counter-example Illustration	• Includes *potential pivotal counterexample #2* o The counterexample falls within the *conventional example space for validation*, which is considered to be students' new potential example space for validation	• Students experience *cognitive conflict #2.* Corresponding *developmental progressions*: o Students progress to the *conventional example space for validation* and recognize empirical arguments as insecure methods for validation (*non-empirical justification scheme*) o Students feel an intellectual need to learn about conventional (secure) methods for validation

Note: The classroom community's learning trajectory represents the trajectory that the class as a *community of learners* was anticipated to followed, as this was reflected in the community's public/collective work. Individual students' hypothetical learning trajectories may deviate from the community's hypothetical learning trajectory.

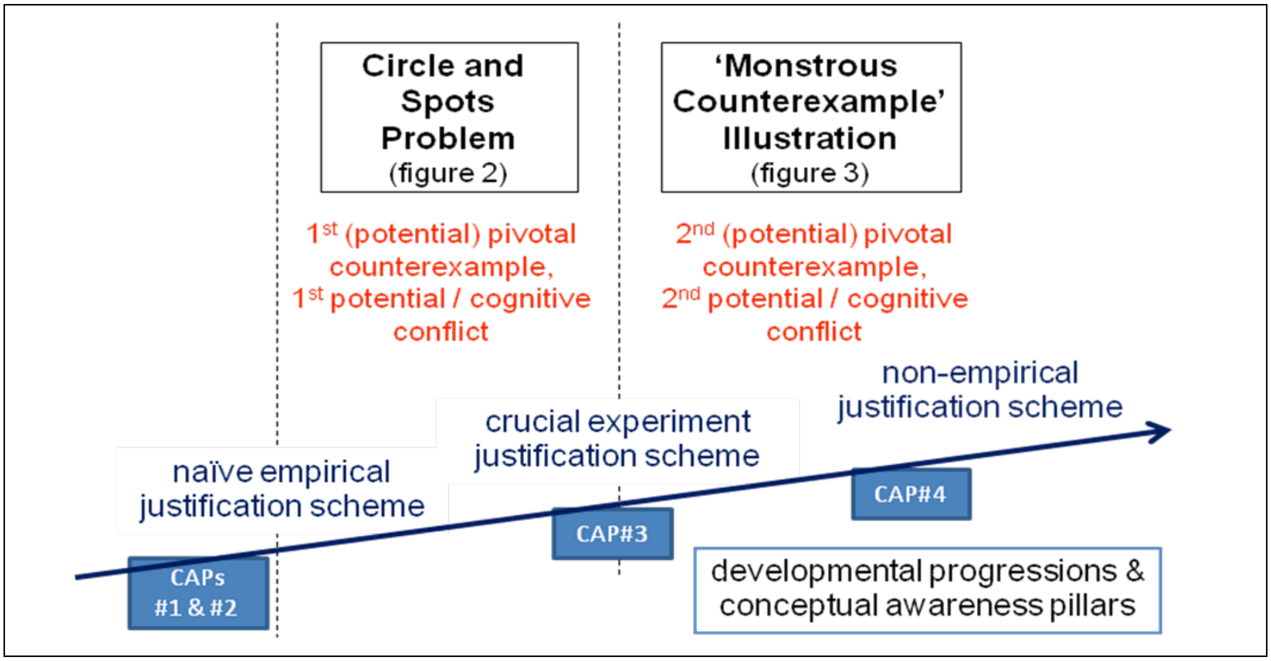

Figure 1. An overview of key aspects of the focal instructional sequence.

Place different numbers of spots around a circle and join each pair of spots by straight lines. Explore a possible relation between the number of spots and the greatest number of non-overlapping regions into which the circle can be divided by this means.

When there are 15 spots around the circle, is there an easy way to tell for sure what is the greatest number of non-overlapping regions into which the circle can be divided?

Figure 2. The Circles and Spots Problem (adapted from Mason et al. [1982]).

Consider the following statement:

> The expression $1+1141n^2$ (where n is a natural number) *never* gives a square number.

People used computers to check this expression and found out that it does ***not*** give a square number for any natural number from 1 to 30,693,385,322,765,657,197,397,207.

BUT

It *gives* a square number for the next natural number!!!

Figure 3. The 'Monstrous Counterexample' Illustration (adapted from Davis [1981]).