Propositions Posed under a Proof without its Proposition

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A proof without its corresponding propositions is given, and students are asked to think what this proof can prove. This study investigates how undergraduate students of mathematics and 9th graders respond to this kind of task. It is found that good proof writers are not necessarily good at this proposition-posing task. The categories of students' responses and the interview scheme developed and used herein should be useful to teachers and other researchers interested in the teaching and learning of proofs with the perspective of reading comprehension. We will discuss students' responses based on three different levels of text processing while reading.

Introduction

> "The mere memorizing of a demonstration in geometry has about the same educational value as the memorizing of a page from the city directory. And yet it must be admitted that a very large number of our pupils do study mathematics in just this way. There can be no doubt that the fault lies with the teaching." John Wesley Young, (1925).

Young's argument had evidenced by some studies. For example, Knuth (2002) found that teachers tended to have a narrow view of proof, which is a topic of study rather than as a tool for communicating and studying mathematics. The majority of teachers in Knuth's study questioned the appropriateness of learning proof for all students. This view contradicts a principle for school mathematics, "reasoning and proof should be a consistent part of students' mathematical experiences in pre kindergarten through grade 12" (NCTM, 2000). Furthermore, Knuth summarized five roles of proof, which included to verify that a statement is true, to explain why a statement is true, to communicate mathematical knowledge, to discover or create new mathematics, or to systematize statements into an axiomatic system, and then found teachers mentioned the various roles of proof except systematizing statements into an axiomatic system. Teachers' views are related to their teaching materials, e.g. textbooks. If researchers can design teaching materials for experiencing and understanding the roles of proof, teachers will have resources to make their efforts to change.

Although researchers agreed with the important roles of informal proofs for learning mathematics, limiting students' experiences with proof to informal methods was not suitable for understanding the nature of proof (Fawcett, 1938; Hanna, 1995).

> "If we consider the nature of a deductive proof, we recognize at once that there must be a hypothesis. It is clear, then, that the starting point of any mathematical science must be a set of one or more propositions which remain entirely unproved." John Wesley Young, (1925).

Referring to Young's statements, some propositions were assumed known and some terms must be undefined (Tarski, 1959). Furthermore, one statement would have different operational statuses (Duval, 1998), and more than 60% of 9th graders had difficulty in distinguishing the logic status from the epistemic status (Lin and Yang, 2007). The need for understanding the axiomatic method is not well arranged by teaching materials for students and teachers. Therefore, this method may be understood as ritual in students' proof scheme (Harel & Sowder, 1998) or may be ignored by teachers (Knuth, 2002).

In sum, learning to write and read proofs is a challenging task for most students. Even if students can write down a formal proof, they may just keep it in mind rather than understand why and how to construct a formal proof (Selden & Selden, 2003). The difficulty in learning proofs brings teachers to a dilemma of skimming this topics or investigating effective teaching approaches, while the items of writing proofs are excluded in the entrance examinations. For providing teachers teaching materials which can really enhance students' understanding of proofs, a task is designed on the basis of problem posing (ref. Silver, 1994; 1997). Problem-posing tasks provide opportunities for students to posing their own problems. Students could formulate problems from given situations or problems. There are no absolutely correct or wrong answers for the problem-posing tasks. While this idea is adapted to mathematical proof, figures or propositions could be substituted for situations or problems (e.g. Lin & Yang, 2002). Students could be make conjectures from false propositions (Lin & Wu, 2005), and working within a graphic milieu could lead students to produce, discuss and test the validity of mathematical statements and theorems (Bloch, 2003). These research results showed that conjecture-posing tasks are beneficial for students' understanding of mathematical proofs and competencies of reasoning.

A conjecture could be absolutely true or false, and the goal of verifying it is subsequently emerged. However, students may hesitate to produce a conjecture that they are not sure if it is correct (Yang, 2007). How could students be encouraged to produce a plausible proposition and smoothly to discuss the validity of this proposition? This paper suggested and evaluated a task for eliminating the hesitation, where students could produce propositions based on their reading comprehension. A proof without its corresponding propositions is given, and students are asked to think what this proof can prove. The research questions include which propositions are posed by undergraduates of mathematics and 9th graders and how they read proof.

Method

Forty seven undergraduates of mathematics who were taking a course of introduction to mathematics and thirty 9th graders were the subjects of this study. The undergraduates were asked to answer the proposition-posing task, shown in figure 1, in a mid-term examination, after such topics as modeling, set, logic and proof method were introduced. A similar task, shown in figure 2, was presented to the 9th graders in mathematics class. They were guided to complete this task by their mathematics teacher. Three undergraduates and three 9th graders were interviewed after the characteristics of their answers were analyzed.

What does the following proof prove?

Assume that $x > 0$, $x < y$, and $\sqrt{x} \geqq \sqrt{y}$.

Case 1. Assume that $\sqrt{x} = \sqrt{y}$. Then $x=(\sqrt{x})^2=(\sqrt{y})^2=y$ and hence, by **A1**, we cannot have $x < y$.

Case 2. Assume that $\sqrt{x} > \sqrt{y}$. Since $\sqrt{x} > 0$, by **A2**, we obtain $\sqrt{x}\sqrt{x} > \sqrt{x}\sqrt{y}$. Also, by **A2**, $\sqrt{x}\sqrt{y} > \sqrt{y}\sqrt{y}$. Now by **A3**, $\sqrt{x}\sqrt{x} > \sqrt{y}\sqrt{y}$. Hence, by algebra, $x > y$. But by **A1**, this is contrary to the assumption that $x < y$.

A1: "*For any real numbers x and y, exactly one of x < y, y < x, and x=y is true.*"
A2: "*If x<y and 0<z, then xz < yz.*"
A3: "If x<y and y<z, then x<z."

Fig. 1. Proof without its proposition for undergraduates of mathematics.

Learning Proof: What does the following proof prove? (_______ property)

Referred to this attached figure,

(1) In □OPC and □OPD,
□□_____=□_____ (given)
_____=_____ (common side)
□_____=□_____ (given)
□□OPC≅□OPD (_____ Property)

(2) By □OPC≅□OPD,
We know _____=_____.
(the corresponding sides are equal)

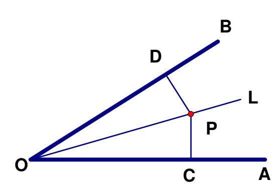

Fig. 2. Proof without its proposition for 9th graders.

While reading texts, readers' intentional goals strongly affected them to select and process information of texts, which are part of reading strategies. Moreover, readers evaluate their reading comprehension to judge if their reading goals are achieved. Accordingly, the interview scheme was designed with respect to three phrases-readers' intentions, reading strategies and readers' evaluation of their reading comprehension. The questions used to interview students included (a) when and why do you stop reading information in the worksheet? (b) what do you do while stopping reading? (c) when and how do you read this proof? (d) how do you know if yon have or have not understood this proof and strike out this answer?

Stine-Morrow, Miller and Hertzog (2006) proposed a model of self-regulated language processing on the basis that readers allocate effort to the computations producing comprehension so as to create a representation that is good enough for their current processing goals. The language representation is constructed at the levels of the word, the textbase, and the discourse. The goals including cognitive, emotional and social goals influence readers' attention allocated to the levels of language processing which constitute reference values. Readers' perceived comprehension of each level is compared with reference values for judging if a representation is good enough. The key functions of the SRLP model are illustrated in Figure 3. This model would be used to elaborate our findings.

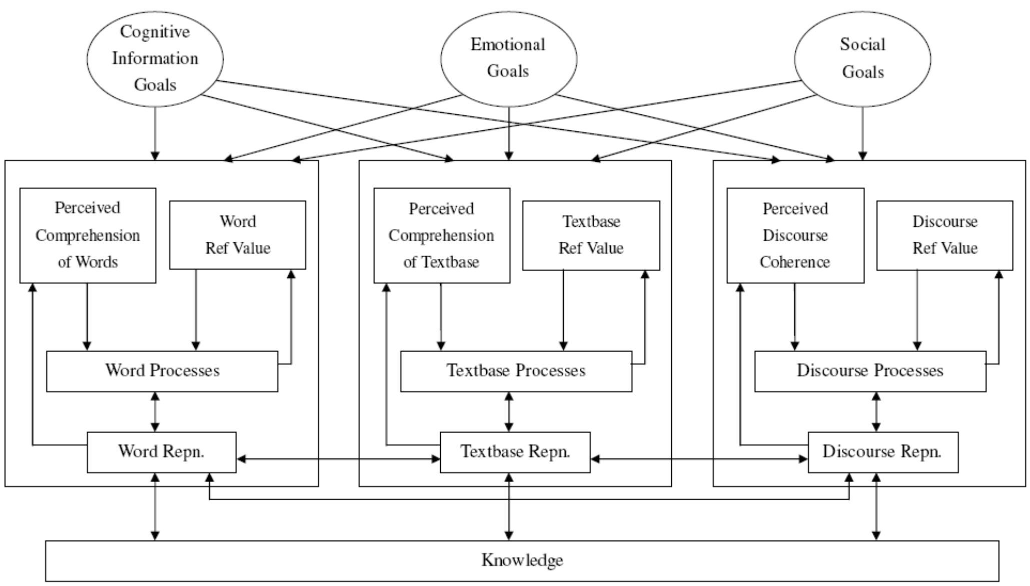

Fig.3 Simplified model of self-regulated language processing. Ref = reference; Repn = representation.

Result

The characteristics of propositions posed by undergraduates of mathematics could be classified in to the five categories of premises-conclusions (13/47), only premises or conclusions (2/47), applied arguments (5/47), only proof method (1/47) and description of proof (10/47). Less than half of the undergraduates in this study posed a complete proposition with premise-conclusion as the answer to the question of what this proof could prove. It surprised us that some undergraduates did not distinguish the conclusion $\sqrt{x}$□$\sqrt{y}$ from premises, $x > 0$ and $x < y$, although some of them had identified this proof method as proof by contradiction, or some of them had posed a complete proposition. After analyzing the interview data, we found some undergraduates might reject to believe this proof and criticize the rationale of this question, what this proof could prove. Some stopped reading while they felt that keeping reading could not improve their comprehension, or they had found the answer even if they had not comprehended the whole proof. Some might review this proof after proposing a proposition or two competing propositions for verifying their answers, and some might argue that the number of the correct answers is not only one.

The characteristics of propositions posed by 9th graders could be classified into the categories of premises-conclusions (17/30), only premises or conclusions (5/30), and description of proof (1/30). It seems that half of the 9th graders could identify premises and conclusions which compose properties. However, the premises or conclusions posed by them were not necessarily consistent with formal or expected answers. For example, $\angle DOP = \angle COP$ or □OPC ≅ □OPD rather than a point on an angle bisector was one premise, and □OPC ≅ □OPD, $\overline{OD} = \overline{OC}$ or $\overline{DC} = \overline{DC}$ rather than that the distances from a point on an angle bisector to both sides of this angle are equal was one conclusion. We also found that 9th graders could use multiple reading strategies like overall review, backward inference, labeling, and switching. But, some students stopped reading because they were not sure the correctness of both proof and proposition, and then they could not complete this task.

Regarding both undergraduates and 9th graders, it is found that good proof writers are not necessarily good at this proposition-posing task, and that some preferred learning how to write proof to learning how to read proof, but some opposed to this preference. Some of undergraduates said that they did not know the starting and ending points in this task besides they were unfamiliar with this kind of tasks. Most of undergraduates appreciated the value of learning how to read proof, but not most of them appreciated the value of reading a proof without a proposition.

Discussion

Under the circumstance of giving a proof without its proposition and asking students to answer the question of what this proof could prove, the research questions include which propositions are posed by undergraduates of mathematics and 9th graders and how they read proof. The answers posed by students could be classified into the five categories premises-conclusions, only premises or conclusions, applied arguments, only proof method, and description of proof in addition to no or nonsense response. However, alternative premises or conclusions posed by them may be correct or incorrect. 9th graders posed premise and conclusion specific to a figure rather than a general description of a concept or a property. Undergraduates posed premise and conclusion but mixed their logical statuses (ref. Yang & Lin, 2008).

At word process, figural codes were helpful while students shifted verbal codes to the attached figure or caught the meaning of verbal codes according to figural codes. At textbase process, the values of a concept, logical or epistemic value (Duval, 2002), disturbed students' comprehension (see also Yang & Lin, 2008). Moreover, students focused on the epistemic value to evaluate their understanding of proof. At discourse process, some undergraduates identified proof methods by the word of contradiction; however, they still failed to pose a correct proposition. This failure may be resulted from their negative emotions stimulated by the context of examination in addition to misunderstanding of the logical statues of some statements.

What is most disappointing is that not all undergraduates of mathematics appreciate the value of this task for learning how to read proof. Criticizing this proof will cause students not only to be unable to pose a plausible proposition but also to repel the idea of reading to learn the nature of proof. Students might justify their propositions by checking the consistency between this proposition and its proof or by proposing alternative propositions before checking it. Of course, there were still students who did not justify their propositions.

Reading strategies needed to validate proofs often receive little attention in mathematics classroom (Selden & Selden, 2002; 2004). Nonetheless, reading strategies used by our students seemed multiple and diverse under this proposition-posing task. Another factor influencing students' validation of proof and its proposition is the reading goal. The undergraduates were asked to complete this task in a mid-term examination, and the 9th graders were asked to complete this task in a class of learning proof. The different goals may result in negative emotions of some undergraduates and positive emotions of some 9th graders while discussing this task with them. Furthermore, 9 graders tried to pose different propositions validated by right corresponding proof.

This proposition-posing task could be designed to motivate students' meaningful learning of understanding proof. However, this task should be modified to test students' reading comprehension of proof because the single question of what this proof proves could not well evaluate multiple facets of reading comprehension of proof. The teaching experiments of extending this task and integrating it into the secondary geometry class of 9th graders had been done, and the effect of teaching is being analyzed in terms of reading comprehension, construction and geometrical thinking.

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