Puzzles and Proofs:

From Informal to Formal Arguments

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Abstract

This article addresses the importance for teachers to incorporate logic and mathematical puzzles into their lessons. The excitement and motivation these problems create can serve as catalysts to understanding and mastering the concept of proof. Some recommendations are included on how to accomplish this goal.

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Generations of mathematicians and scientists thank Martin Gardner and his ingenious puzzles for having aroused their interest in the mathematical sciences. Many also thank Raymond Smullyan for his extraordinary puzzle-making ability that renders profound results in mathematical logic intelligible to an educated general audience. Moreover, Russian teachers have used Kordemsky's puzzles to breed generations of students in their mathematical circles. But what's the fuss about puzzles? That's not real mathematics. Or is it?

Mathematical and logic puzzles are ideal ways to stimulate the creativity of students, as well as effective means to foster their deductive reasoning abilities. Furthermore, the excitement and motivation these problems create can serve as catalysts to understanding and mastering the concept of proof. After all, conjectures are mathematical puzzles that when solved become theorems. Hence, it is important that teachers – from elementary school to college – learn to incorporate puzzles into their classrooms.

Informal Arguments and Puzzles

Informal arguments are what Polya referred to as “plausible reasoning” (Polya 1954). “This is the kind of reasoning on which [a serious student of mathematics'] creative work will depend” (Polya, p. vi). Fostering this type of reasoning and stimulating students' creative talents are facilitated by identifying problems that can be approached by concrete reenactment. However, “[i]f we wish to talk about mathematics in a way that includes acts of creativity and understanding, then we must be prepared to adopt a different point of view from the one in most books about mathematics and science” (Byers, p.3). This approach emphasizes the dynamic nature of mathematics, which erupts more drastically if the students are allowed to discuss their ideas among themselves as they try to solve the problems.

The idea is not to "make math fun," but to reveal what mathematics is: an intellectually stimulating and gratifying adventure. Consider these slightly modified problems, taken from Gardner (1994), Vakil (1996), and Kordemsky (1992), respectively.

1. *Jim and Joe run a 100-meter race. Jim wins by 10 meters. To even up the race, they decide to run again, but this time Jim begins 10 meters behind the start line. Assuming they run at the same constant speed as before, who wins?*

The almost instinctive answer students give to this problem is that they tie. When prompted to substantiate their claim, it is not unusual to hear, "Well, it's obvious." If encouraged to make a diagram, they might realize their mistake.

Jim wins the first race, covering 100 meters in the same amount of time Joe covers 90. Hence, in the second race, just after Joe reaches 90 meters, Jim will pass him, having covered 100 meters. However, there are still 10 meters left for the run. Since Jim is faster, he will again win the race.

2. *The lockers at Mathematical Sciences High School are numbered 1 through 1000; one for each student. The students have been arguing about which color to paint their lockers and decide that each student will have a turn to paint them as he wishes. It so happens that the first student paints all the lockers yellow. The second student paints all the even numbered lockers blue. The third student changes the color of the third locker from blue to yellow or from yellow to blue, and then moves three lockers down the hall and repeats this pattern until he reaches or passes the last locker. The fourth student runs down the*

hall and does the same for those lockers whose numbers are multiples of 4. In this way the students go on painting the lockers until the last student has her share. At the end of the painting rampage, which lockers are painted yellow and which ones are painted blue?

This problem is much simpler than it appears. It is overwhelming for most high school students, although even younger students with some knowledge of arithmetic and proper guidance can solve it.

Before presenting this problem, it may be useful to talk about prime factorization and divisibility rules, just to get students in an appropriate thinking mood. The teacher may suggest considering a simpler case that preserves the structure of the original problem. For instance, instead of 1000 lockers, consider ten of them.

Yellow	Blue	Blue	**Yellow**	Blue	Blue	Blue	Blue	**Yellow**	Blue
1	2	3	**4**	5	6	7	8	**9**	10

In case that is not enough to help students make a conjecture, they may try 20.

Y	B	B	**Y**	B	B	B	B	**Y**	B	B	B	B	B	B	**Y**	B	B	B	B
1	2	3	**4**	5	6	7	8	**9**	10	11	12	13	14	15	**16**	17	18	19	20

Now the pattern should be clear. The yellow lockers are 1, 4, 9, and 16, which suggests that the perfect squares are the only yellow lockers. Why would that be? Well, the perfect squares are the only numbers that have an odd number of factors, hence those lockers will get an odd number of layers of paint, whereas the rest will get an even number.

3. *A nonstop train leaves New York for Chicago at 60 miles per hour. Another nonstop train leaves Chicago for New York at 40 miles per hour. How far apart are the trains one hour before they pass each other?*

Students often claim the problem is ill-posed for not stating the distance between the two cities. The teacher may ask them to explain why they think this information is relevant. However, the trains have a combined speed of 100 miles per hour, hence, the trains will be exactly 100 miles apart before passing each other. (Of course, the cities must be at least 100 miles apart, which Chicago and New York certainly are.)

Puzzles and Proofs: The connection

Formal proofs in mathematics are similar to solutions of mathematical puzzles. There are important differences, like the complexity of the technical and historical context of mathematics. Yet, if students learn to provide articulate and detailed responses to carefully selected puzzles, they will be better equipped to master the intricacies of mathematical reasoning. After all, although remembering axioms, definitions, and previous results is necessary to prove theorems, the most important component in internalizing the concept of proof is to understand what deductive reasoning is. It is the responsibility of teachers to help students recognize the connection between solving a problem and constructing a proof by demanding detailed explanations of their reasoning, instead of encouraging complacency after finding an answer. As a bonus, the informal setting of puzzles can help teachers identify mathematical talent that may otherwise go unnoticed.

Incorporating Puzzles into the Curriculum

It is incumbent upon mathematics teachers to provide opportunities for students to reveal their talents by eliciting creative solutions to challenging problems. As a general principle, teachers should strive to present mathematics as an interesting and accessible intellectual game, rather than as a set of rules and procedures of practical necessity. In order to present mathematics as a living field, teachers must not be afraid to digress to explore an idea or comment offered by students or to pose questions for which answers may not be accessible in an average class.

Puzzles come in all sorts of forms and contexts, and can fit the curriculum of any classroom at any level. Different solutions to the same problem must be welcomed and encouraged. Moreover, teachers should select problems where students can apply similar techniques as those used to solve another problem.

Teacher Workshops

Ideally, workshops for teachers should involve participants in the same kind of activities recommended for the students they teach. Hence, workshops should focus on helping teachers understand the subject matter they will teach, giving specific and general ideas on how to implement curricular changes. In addition, the workshops should contain a component on creating puzzles, which often requires more ingenuity than solving them.

Four workshop sessions a semester of three to four hours each, might be enough to help teachers feel comfortable with adopting these ideas in their classrooms. Workshops can begin with intriguing puzzles for small-group solution by participating teachers. Groups can report their solution attempts accompanied by discussion and informal evaluation. Workshop leaders should be able to discern and articulate several problem solving strategies. A problem collection that includes alternative solutions as well as grade-level and curricular recommendations for their use is a valuable resource that can be delegated to the workshop participants.

Participation in workshops requires that teachers agree to introduce at least one puzzle weekly in every course they teach, and to select, display, and analyze responses to three puzzles during the weeks following their first workshop session. Participants would prepare and submit case-studies based upon their students' responses to the problems for review by workshop faculty. The workshop faculty should distribute a newsletter based upon the case-studies submitted by participants. Moreover, the newsletter would include new problems and additional resources to aid teachers in implementing these curricular changes. Similar activities are recommended for workshops to be conducted by school themselves.

Conclusion

Some detractors may argue that incorporating puzzles in their lessons will take time away from the subject matter. However, mathematical and logic puzzles foster an intellectually stimulating environment that can not only help create a better foundation on which to build the structure of mathematical knowledge, but advance a culture that values mathematical ideas. This should be the goal of any mathematics education agenda.

References

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