THE DESIGN OF MULTIPLE REPRESENTATION TASKS TO FOSTER CONCEPTUAL DEVELOPMENT

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Introduction

In this paper I outline a number of research-based principles that are used in the design of learning experiences that are intended to foster conceptual understanding in mathematics. These are the illustrated using one particular type of task that I have used extensively with students in secondary and adult education; that of sorting *multiple representations*. I refer to learning *experiences*, because the task itself is only one component of the design. Close attention is also paid to the role of the teacher in creating an appropriate climate for learning to take place.

Theoretical framework

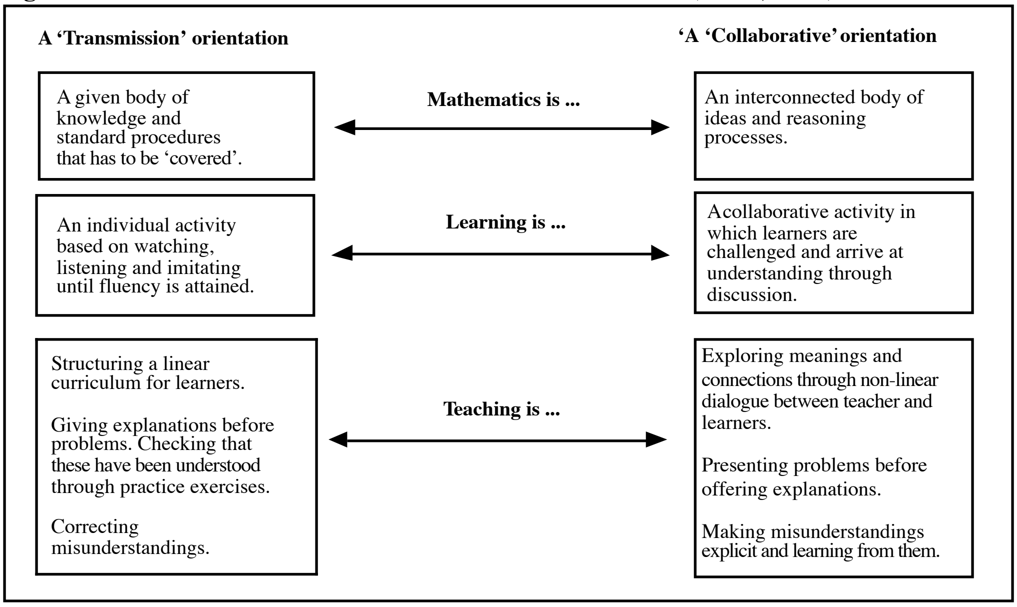
My own designs for novel mathematical tasks are based on a three-way analysis of: the purposes they are intended to serve; the learning theories related to those purposes; and the empirically tested principles for design that emerge from these theories. Clearly, different learning theories apply when different learning outcomes are desired. A full analysis of these purposes, theories and principles are available in Swan (2006a).

There are perhaps five distinct purposes for learning mathematics: (i) developing fluency when recalling facts and performing skills; (ii) interpreting concepts and representations; (iii) developing strategies for investigation and problem solving; (iv) awareness of the nature and values of the educational system and (v) an appreciation of the power of mathematics in society (e.g. (Cockcroft, 1982; NCTM, 1988; Stigler et al., 1999; Stigler & Hiebert, 1999). In the examples described below, my purpose is conceptual.

The learning theories I adopt for this purpose are derived from the social constructivists: concepts are co-created as language and symbols are appropriated and internalised (Bakhtin, 1981; Vygotsky, 1996). Collaborative discussion is therefore essential. Other relevant design principles include the importance of focusing directly on significant conceptual obstacles (Bell, 1993; Wigley, 1994); eliciting, confronting and building on the knowledge students already have (Black & Wiliam, 1998); carefully juxtaposing questions and stimuli so as to produce surprise, tension and cognitive conflict that may be resolved through reflection and discussion (Bell, 1993; Bell et al., 1985; Brousseau, 1997); using tasks that are accessible, extendable, encourage decision-making, creativity and higher order questioning (Ahmed, 1987); using multiple representations to create bridges between concepts (Askew et al., 1997); and using tasks that allow students to shift roles and explain and teach one another (Bell et al., 1993b).

These principles challenge many teachers' existing orientations towards mathematics, learning and teaching as outlined in Figure 1, below. A 'transmission' orientation in which explanations, examples and exercises dominate must give way to a more collaborative orientation in which students work together on 'connected', 'challenging' tasks; tasks that emphasise the interconnected nature of mathematics and confront common difficulties. This model of learning should not be confused with that of 'discovery' teaching, where the teacher simply presents tasks and expects students to explore and discover the ideas for themselves. Here, the teacher's role includes: assessing students and making constructive use of prior knowledge; making the purposes of activities clear; challenging students through effective, probing questions; managing small group and whole class discussions; encouraging the discussion of alternative viewpoints; drawing out the important ideas in each lesson; and helping students to make connections between their ideas.

Figure 1: From a transmission to a collaborative orientation (Swan, 2005)



Types of task

In coming to understand a concept, a student must single it out and bring it to the forefront of attention (identify); notice similarities and differences between this concept and other similar ones (discriminate); identify general properties of the concept in particular cases of it (generalise) and begins to perceive a unifying principle (synthesise) (Sierpinska, 1994). Working with teachers, we have developed five task 'types' that encourage these processes. These are summarised below, in Table 1. The first of these is discussed Zaslavsky's presentation to this conference. The second is the main focus of this paper.

Table 1: Five task 'types' that encourage concept development

Classifying mathematical objects	Students devise their own classifications for mathematical objects, and/or apply classifications devised by others. In doing this, they learn to discriminate carefully and recognise the properties of objects. They also develop mathematical language and definitions. The objects might be anything from shapes to quadratic equations.
Interpreting multiple representations	Students work together matching cards that show different representations of the same mathematical idea. They draw links between representations and develop new mental images for concepts.
Evaluating mathematical statements	Students decide whether given statements are always, sometimes or never true. They are encouraged to develop mathematical arguments and justifications, and devise examples and counterexamples to defend their reasoning. For example, is the following statement always, sometimes or never true? If sometimes, then when? "Jim got a 15% pay rise. Jane got a 10% pay rise. So Jim's pay rise was greater than Jane's."
Creating problems	Students are asked to ceate problems for other students to solve. When the 'solver' becomes stuck, the problem 'creators' take on the role of teacher and explainer. In these activities, the 'doing' and 'undoing' processes of mathematics are exemplified. For example, one partner may create an equation, then the other tries to solve it.
Analysing reasoning and solutions	Students compare different methods for doing a problem, organise solutions and/ or diagnose the causes of errors in solutions. They begin to recognise that there are alternative pathways through a problem, and develop their own chains of reasoning.

In addition to the tasks, we also pay particular attention to the ways in which shared resources such as posters, mini-whiteboards, and computer software may be used to encourage collaborative learning. (The small physical size of most books (and workbooks) can be an obstacle to effective collaboration).

Posters are often used in schools and colleges to display the finished, polished work of students. In our work, however, we use them to promote collaborative thinking. The posters are not produced at the end of the learning activity; they are used throughout to display the group's reasoning.

Mini-whiteboards are small erasable plastic boards that students write on in marker pens. They are a powerful pedagaogical aid to whole class discussion for several reasons: in whole class discussion, they allow the teacher to ask new kinds of open question (typically beginning: 'Show me....'); students can all respond together so that the teacher sees what each student thinks; they also encourage students to share private, rough working that may be quickly erased.

Tasks that involve interpreting multiple representations

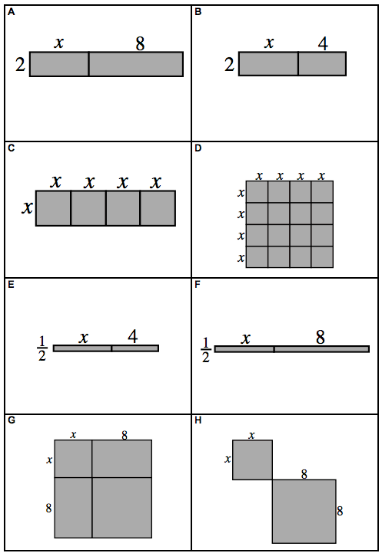
In the remainder of this paper, I describe just one of the above task types, "*Interpreting multiple representations*" and show how it relates to the learning principles described above. This task involves students sorting cards that show different representations of mathematical objects into sets so that each set has equivalent meaning. These objects may be words, pictures, numbers, symbols or charts of almost any kind. Figure 2 offers two typical card sets that have been used.

Figure 2: Two examples of card sets

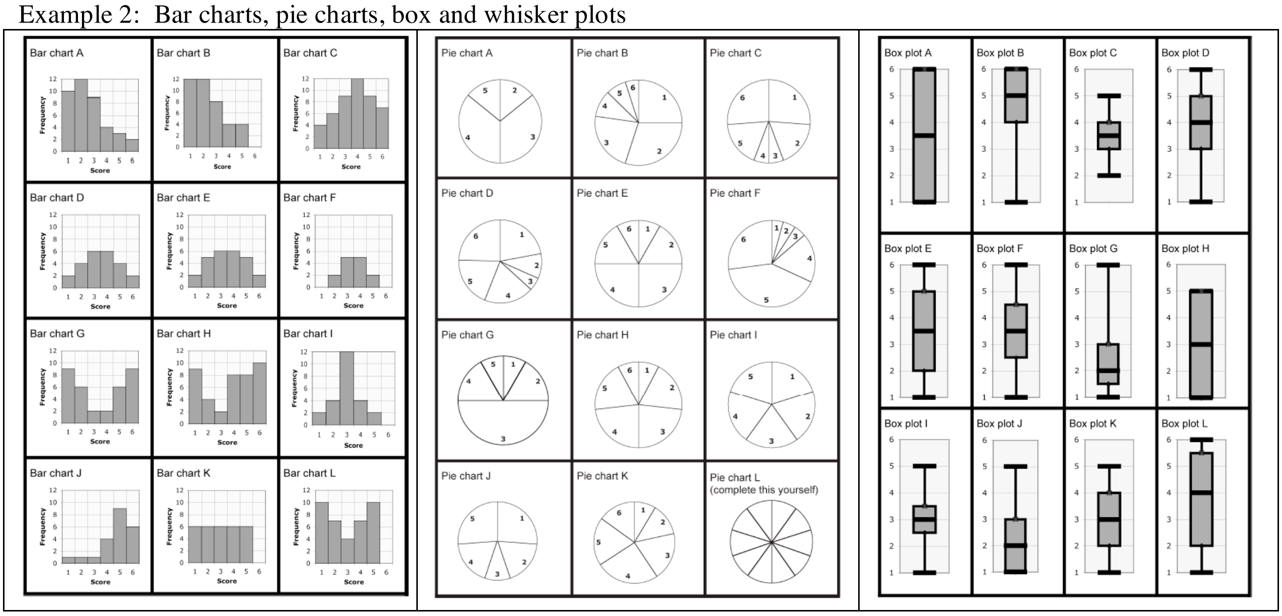
Example 1: Words, algebra expressions, areas

A Multiply x by two, then add eight.	B Multiply x by four, then square the answer
C Add eight to x then multiply by two.	D Add eight to x then divide by two
E Add four to x then multiply by two.	F Add eight to x then square the answer
G Multiply x by two then add sixteen	H Divide x by two then add four
I Square x then add sixty four	J Square x then multiply by four
K Add four to x then divide by two.	L Divide x by two then add two
M Square x then multiply by sixteen	N

A $2(x+8)$	B $4x^2$
C $2x+8$	D $2x+16$
E $2(x+4)$	F $(4x)^2$
G $16x^2$	H $(x+8)^2$
I $x^2+16x+64$	J $\frac{x+4}{2}$
K	L x^2+8^2
M $\frac{x+8}{2}$	N $\frac{x}{2}+4$



Example 2: Bar charts, pie charts, box and whisker plots



Students take turns to match pairs of cards, explain their relationship and place them face up on the table. Other students challenge or seek clarification. The card sets are designed so that students have to discriminate carefully and confront common conceptual obstacles. They are also designed so that matching is not necessarily 1 to 1. This encourages connections and relationships to be explored. In addition, sets are usually incomplete so that students have to construct new examples. The intention is that, through discussion, students articulate their own interpretations for cards and with the assistance of the teacher begin to form generalisations. Students present their results to the whole class by pasting card sets onto posters and annotating these with explanations. A concluding plenary discussion is used to identify significant results and promote generalisation. This 'institutionalises' learning (Brousseau, 1997).

In designing such a set of cards, one has to carefully consider:

- The affordances and limitations of each representation;
- How the cards will be used to expose common misconceptions;
- The order in which the representations are presented to maximise conflict and discussion;
- The criteria that students use to match cards, and how we can minimise superficial matching;
- The degree to which sets of cards can open up the possibility for generalisation.

Example: Interpreting Percentage increase and decrease.

Before using this activity, we assume that students have already engaged in activities that involve matching equivalent representations of decimals, fractions, percentage and area cards.

Students, in small groups, are given Money cards ('states') and Percentage cards ('changes'). Students are invited to take turns at choosing cards and placing these so that between two states there are appropriate changes (Figure 3). Typically, they make the mistake of pairing an increase of 50% with a decrease of 50%, and so on. (Notice that the design of the cards must permit this possibility). Such errors are not commented on at this stage. This part is intended to expose misconceptions such as: n% increase followed by an n% decrease results in no change.

Groups are now issued with a packs of 'verbal descriptions' (Figure 4). Students combine these cards with those already on the table. This encourages them to translate between common percentages and fractions (e.g. 25%, one quarter).

Figure 3: A correct matching of the percentage increase and decrease cards.

They are then issued with the 'decimal muliplier' cards and check that they are correctly positioned with calculators. This often provides conflict as students realise that their earlier positioning was incorrect.

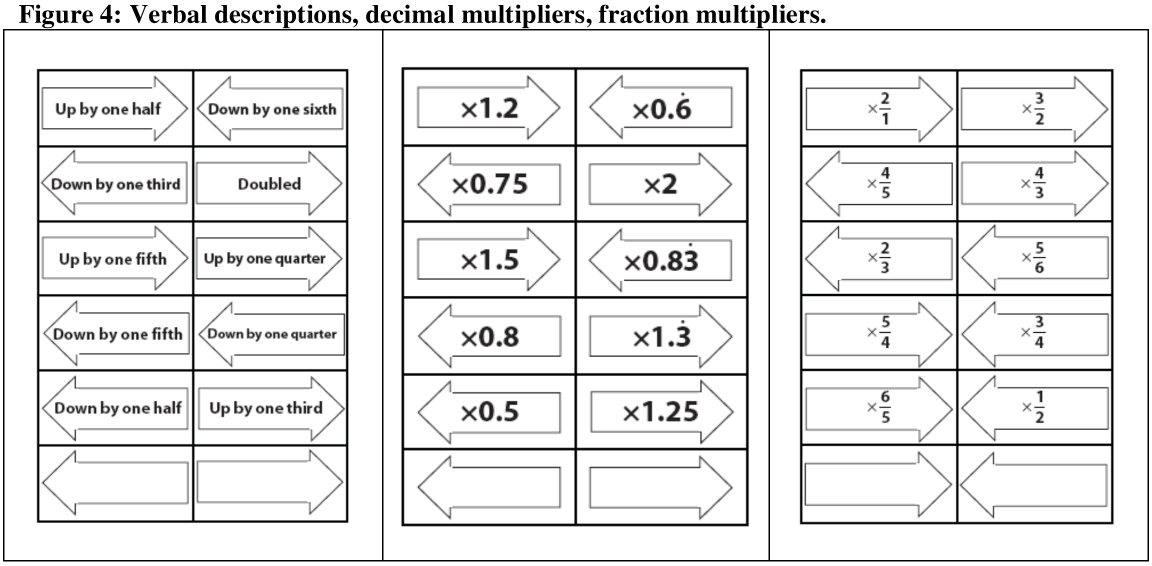
They now begin to make links between percentage increases and decimal multipliers (Up by 50%, multiply by 1.5) and between fractions and decimals (e.g. "up by one half", "multiply by 1.5"). Finally, students are given the fraction multiplier cards and further connections are made. Throughout this complex process, students are encouraged to make connections and generalisations. For example, they may notice that there is a clear pattern in the pairs of words that represent inverse functions:

Doubled	Down by one half
Up by one half	Down by one third

Up by one third	Down by one quarter
Up by one quarter	Down by one fifth
Up by one fifth	Down by one sixth

Such patterns demand explanation. Students finally assemble posters from cards and present their findings to the rest of the class. This *gives status* to their own ideas. The teacher encourages students to *extend* and *generalise* their ideas by making small changes to the examples and by explicitly formulating general rules for equivalence. The teacher can, for example, suggest replacing the money cards with geometrical shapes. The teacher's role is thus to recognise and values the important contributions of students, and extend and 'institutionalise' them.

Figure 4: Verbal descriptions, decimal multipliers, fraction multipliers.



Discussion

Multiple representations tasks illustrate the following principles for teaching and learning:

They expose and build on existing interpretations and understandings. These lessons do not begin with explanation. Instead, the teacher asks questions and presents an activity that exposes existing ways of thinking and reasoning. Students are confronted with inconsistencies and alternative interpretations through discussion. Conflicts originate both internally, within the individual and externally, from an individual’s interpretation of another person’s alternative viewpoint. Explanation follows this discussion.

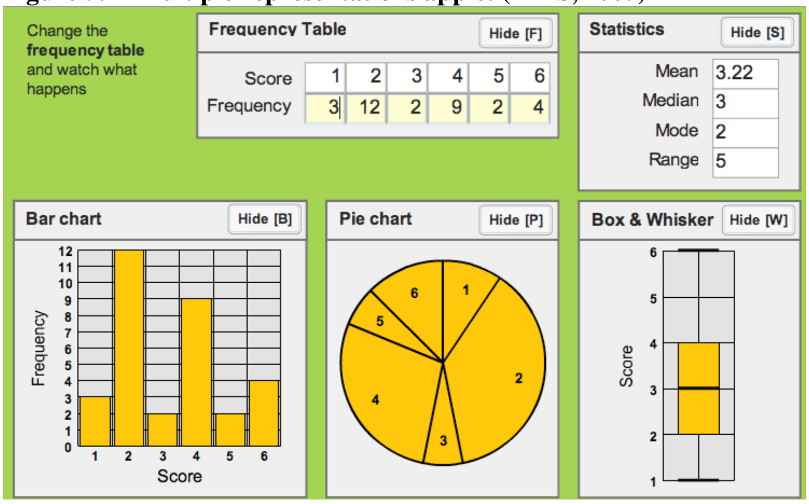
They use cooperative small group work. Interpretations remain mere ‘shadows’ unless they are articulated through language (Vygotsky, 1996). Social interaction is thus centrally important. We find that many of our students have never had much opportunity to put their own understanding into words. The teacher has to try to intervene in a proactive but collaborative non-judgmental manner, challenging students to explain and justify their reasoning.

They use rich, open tasks that have multiple entry points and many pathways. Conceptual frameworks do not develop along pre-determined linear hierarchies. Tasks are designed so as to provide opportunities for students to create their own multiple connections.

They value improved comprehension over the completion of 'products'. Students need periods of ‘stillness’ (not necessarily silence) when ‘production of answers’ gives way to ‘reflecting on alternative methods and meanings’. These are substantial tasks, sometimes taking one hour or more, and progress is slow. Many students may not finish a given task. This can worry some teachers. Teachers are also sometimes concernedover the apparent lack of individual, tangible products: "What is there to assess/take away at the end of a discussion activity?" Both of these issues need addressing in professional development.

They can also use technology to facilitate discussion. Technology can be a powerful resource, particularly for the whole class discussions. Students are able to replicate their sorting and place cards dynamically using interactive whiteboards. Computer applets allow the interactions between each representation to be explored. In the example below, (Figure 5), the frequency table may be varied and the effects observed on other representations. Then, one or more representations may be hidden and their appearance predicted from the infomation available in the remaining ones.

Figure 5: A multiple representations applet (DfES, 2005)



Although these complex activities pose considerable management challenges for teachers, they have been extremely popular with both teachers and students and have strongly featured in three recent projects in which I have been involved (DfES, 2005; NCEE, 2006; NRDC, 2006). The evidence from these suggests the above task types have been popular with students, have improved learning outcomes and have also had an impact on the beliefs and practices of many mathematics teachers (Swain & Swan, 2007; Swan, 2006a, 2006b, 2007).

> *The good thing about this was, instead of like working out of your textbook, you had to use your brain before you could go anywhere else with it. You had to actually sit down and think about it. And when you did think about it you had someone else to help you along with you if you couldn't figure it out for yourself, so if they understood it and you didn't they would help you out with it. If you were doing it out of a textbook you wouldn't get that help. After I did it I found that I used a lot of brain power, but I felt dead clever. Do you know that when you have actually done something and you actually put all your effort into something.. it makes you feel dead clever. I've told all my friends that I have actually done a bit of work in maths. 'Cause I never thought I was any good at maths, but I was alright with that.*
> *(Lauren, a 16-year-old low-attaining student).*

References

Ahmed, A. (1987). *Better Mathematics: A Curriculum Development Study*. London: HMSO.

Askew, M., Brown, M., Rhodes, V., Johnson, D., & Wiliam, D. (1997). *Effective Teachers of Numeracy, Final Report*. London: Kings College.

Bakhtin, M. M. (1981). *The dialogic imagination: Four essays by M.M. Bakhtin* (C. Emerson & M. Holquist, Trans.): University of Texas Press.

Bell, A. (1993). Principles for the Design of Teaching. *Educational Studies in Mathematics, 24*(1), 5-34.

Bell, A., Swan, M., Crust, R., & Shannon, A. (1993b). *Awareness of Learning, Reflection and Transfer in School Mathematics; Report of ESRC Project R000-23-2329*: Shell Centre for Mathematical Education, University of Nottingham.

Bell, A., Swan, M., Onslow, B., Pratt, K., & Purdy, D. (1985). *Diagnostic Teaching for Long Term Learning. Report of ESRC Project HR8491/1*: Shell Centre for Mathematical Education, University of Nottingham.

Black, P., & Wiliam, D. (1998). *Inside the black box : raising standards through classroom assessment*. London: King's College London School of Education 1998.
Brousseau, G. (1997). *Theory of Didactical Situations in Mathematics* (N. Balacheff, M. Cooper, R. Sutherland & V. Warfield, Trans. Vol. 19). Dordrecht: Kluwer.
Cockcroft, W. H. (1982). *Mathematics Counts*. London: HMSO.
DfES. (2005). *Improving Learning in Mathematics*. London: Standards Unit, Teaching and Learning Division.
NCEE. (2006). *America's Choice: Mathematics Navigator*. Washington, US.
NCTM. (1988). NCTM Curriculum and Evaluation Standards for School Mathematics: Responses from the Research Community. *Journal for Research in Mathematics Education, 19*, 338-344.
NRDC. (2006). *Maths4Life: Thinking through Mathematics*. London: DfES.
Sierpinska, A. (1994). *Understanding in Mathematics,* . London: Falmer.
Stigler, J. W., Gonzales, P., Kawanaka, T., Knoll, S., & Serrano, A. (1999). *The TIMSS Videotape Classroom Study: Methods and Findings from an Exploratory Research Project on Eighth-Grade Mathematics Instruction in Germany, Japan, and the United States (NCES 1999-074).* Washington, DC: National Center for Education Statistics.
Stigler, J. W., & Hiebert, J. (1999). *The Teaching Gap* (2 ed.). New York: The Free Press.
Swain, J., & Swan, M. (2007). *Thinking Through Mathematics research report*. London: NRDC.
Swan, M. (2005). *Improving Learning in Mathematics: Challenges and Strategies*. Sheffield: Teaching and Learning Division, Department for Education and Skills Standards Unit.
Swan, M. (2006a). *Collaborative Learning in Mathematics: A Challenge to our Beliefs and Practices*. London: National Institute for Advanced and Continuing Education (NIACE); National Research and Development Centre for Adult Literacy and Numeracy (NRDC).
Swan, M. (2006b). Learning GCSE mathematics through discussion: What are the effects on students? *Journal of Further and Higher Education, 30*(3), 229-241.
Swan, M. (2007). The impact of task-based professional development on teachers' practices and beliefs: A design research study. *Journal of Mathematics Teacher Education, 10*(4-6), 217-237.
Vygotsky, L. (1996). *Thought and Language* (A. Kouzlin, Trans. 9th ed.). Cambridge: Massachusetts Institute of Technology Press.
Wigley, A. (1994). Models for Mathematics Teaching. In A. Bloomfield & T. Harries (Eds.), *Teaching and Learning Mathematics* (pp. 22-25). Derby: Association of Teachers of Mathematics.