ATTENTION TO SIMILARITIES AND DIFFERENCES: A FUNDAMENTAL PRINCIPLE FOR TASK DESIGN AND IMPLEMENTATION IN MATHEMATICS EDUCATION

Orit Zaslavsky
Dept. of Education in Technology & Science
Technion, Haifa, Israel
orit@technion.ac.il

The cognitive activity of classifying is a natural power that helps structuring and clarifying the entities we perceive, enables people to make distinctions, and reduces the amount of memory needed for dealing with new information (Bornstein, 1984; Lakoff, 1987; Reed, 1972). Rosch's (1973, 1975) studies shed light on the vital role of classification processes with respect to human thinking. Associating concepts with categories is at the heart of learning. It requires the identification of similarities and differences between objects along several dimensions, which is fundamental to mathematical thinking. Classification and characterization of different objects according to various criteria may enhance awareness of ways in which they are related to each other (Silver, 1979; English, 1997; Swan, 2007).

I examine tasks for secondary mathematics students that require comparing and/or classifying different mathematical objects or other entities according to various criteria that are open to learners to suggest (Zaslavsky et al, 2003; Zaslavsky & Leikin, 2004). Such tasks are open-ended in nature (in the spirit of Zaslavsky, 1995) and may provide a rich context for eliciting many viewpoints regarding mathematical structures. They are "low risk", as different learners may approach them in different ways, some attending to more immediate features, and others to deep structural ones. They have the potential of drawing attention and raising awareness, generating much discussion on a wide range of issues, including common features of various families of objects, different representations of mathematical objects, and connections between them. The special nature of this family of tasks makes them accessible and applicable to various communities of practice (students, teachers, and teacher educators). They also may be used to identify learners' mathematical thinking and understandings.

In my presentation I will discuss and illustrate 3 generic types of tasks that call for drawing connections and making comparisons between mathematical objects. As Swan (this session) maintains, these kinds of tasks address concept development in the broad sense. They also lend themselves to genuine cooperative learning, organized in small groups, where all members are actively engaged (Leikin & Zaslavsky, 1997).

The three types of tasks are: 1. Sorting Tasks; 2. Considering Alternatives; 3. Compare and Contrast Tasks.

1. <u>Sorting Tasks</u>:

Typically, in a sorting task 20-30 cards are given, and the activity is to sort the cards in as many different ways that make sense to the learner. For each sorting, the group needs to state the criterion by which they sorted the cards and to specify the different categories within the criterion (e.g., if the criterion for sorting is color and the categories could be: red, blue, green and black). The groups are asked to keep a record of the order/sequence of the criteria they chose to apply. Within each group, and then in a whole-class discussions, there are opportunities for sharing with peers and discussing various ways of sorting and for suggesting more ways. The task includes reflection on what learning it facilitated.

The design of such tasks is rather demanding and requires careful consideration of the choice of the specific examples/objects with respect to numerous criteria by which they may be grouped.

Examples of types of objects to sort:

- Mathematical objects/examples, such as: Functions of various families, in various representations; Equations and inequalities; Geometric shapes;
- Mathematical problems;

The objects may be given in one form of representation or in different representations. Matching and grouping equivalent objects in different representations (as in Figure 2 of Swan's paper for this session) can also be seen as a sorting task.

It should be noted that there is a difference if the task is designed with real cards or as a table. With actual cards the task implies that for a particular sorting each card should be classified into one and only one group of cards; however, if the objects are given in a table (e.g., Figure 1) it may raise the option that an object may belong to more than one group. This seemingly minor difference makes a big difference in terms of the kinds of classifications that are suggested.

Figure 1 is an example of a sub-set of 15 cards for a sorting task that was given to 11^{th} grade students. The other half of the full set (another 15 cards) is not disclosed in this paper. During my presentation, the audience will be asked to suggest additional cards that would make this sorting task richer and more powerful in terms of conceptual learning affordances. By experiencing the need to construct and select additional objects for sorting, some of the design principles will be made more explicit.

In general, the set of cards consists of graphs (contrary to Swan's presentation in this session, the form of representation is kept constant). There are several criteria by which these objects may be grouped. The ability to switch from one sorting to another requires changing the lens and seeing the same objects in different ways that is, focusing on different features of the object. In Mason's (1989) term it involves shifting attention that facilitates generalization and abstraction. It can be seen as manifestation of higher order thinking as well as a means for developing it.

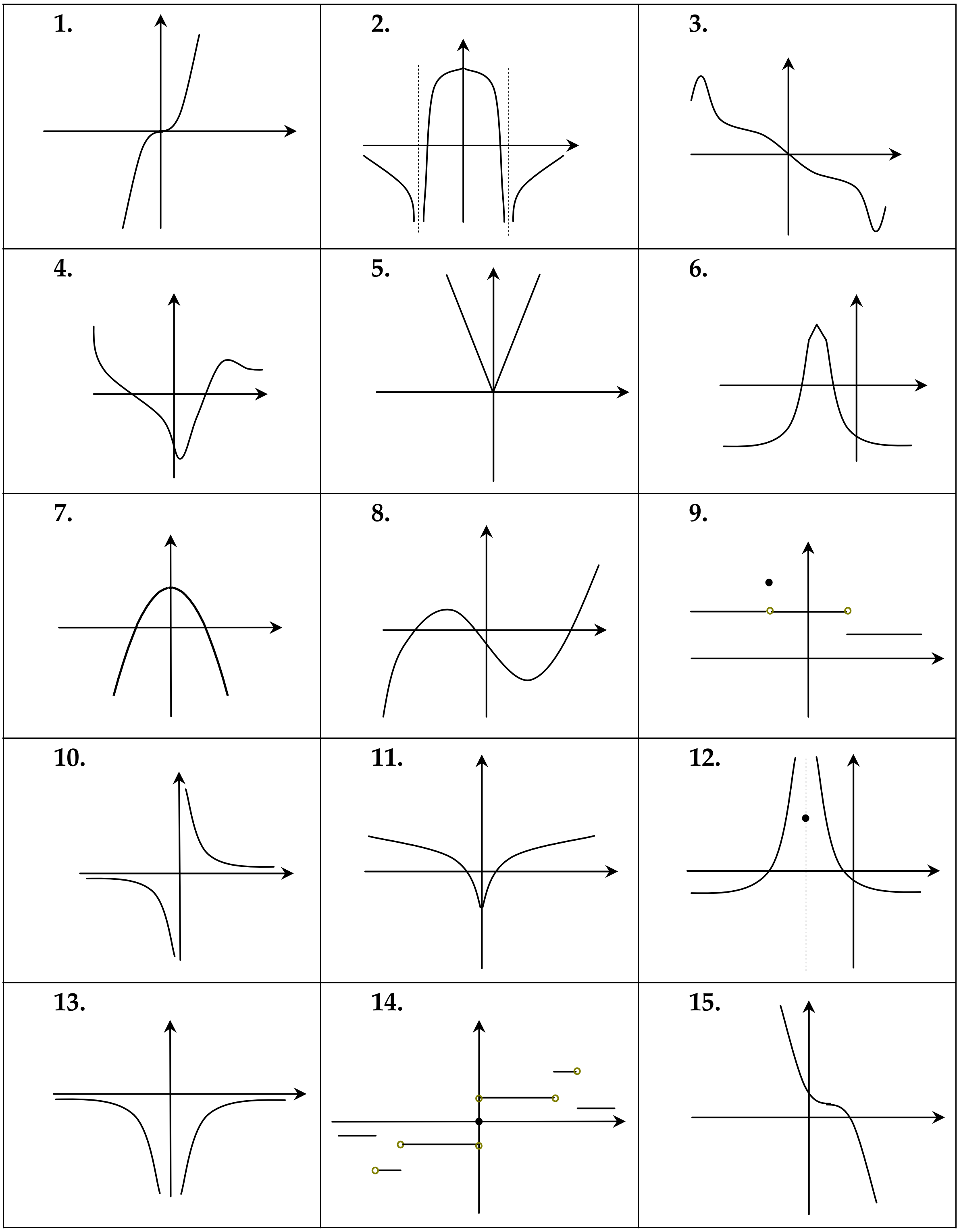

Figure 1: Subset of Cards for a Sorting Task

Assuming that each graph represents a function, some possible criteria for sorting are:

- *Function Parity* - Categories: odd functions (1, 3, 10, 14); even functions (2, 5, 7, 11, 13); functions that are neither odd nor even (4, 6, 8, 9, 12, 15));

- *Symmetry* – Categories: non-symmetrical functions (4, 9); symmetrical functions: with line symmetry (2, 5, 6, 7, 10, 11, 12, 13); with point symmetry (1, 3, 8, 10, 14, 15); with both – line and point symmetry (10). This criterion opens the opportunity to make distinctions between symmetrical functions and odd or even functions. If the task is in the form of actual cards, there is a dilemma in what group to put function 10. This could lead to determining disjoint categories, for example, by distinguishing between functions that have only line symmetry, those that have only point symmetry, and function that have both. Under this categorization, function 10 belongs just in the latter.
- *Continuity* over $\mathbb{R}$ - Categories: continuous functions (1, 3, 4, 5, 6, 7, 8, 11, 15); discontinuous functions (2, 9, 10, 12, 13, 14). The category of discontinuous functions can be further divided according to type of discontinuity (removable vs. irremovable) or number of discontinuity points (in these cards there are discontinuous functions with one discontinuity point (10, 12, 13), two (2, 9), or five (14)).
- *Differentiability* – Categories: functions that appear to be differentiable in every point of their domain (1, 2, 3, 4, 7, 8, 10, 13, 15); functions that appear to have points in their domain for which they are not differentiable (5, 6, 9, 11, 12, 14). This criterion lends itself to an authentic need to make distinctions and examine the logical connections between continuity and differentiability.
- *Asymptotes* – Categories: functions that have at least one asymptote (2, 10, 12, 13); functions that do not have an asymptote (the rest).
- *Monotony* – Categories: monotonic functions, increasing functions (1); decreasing functions (15); non-monotonic functions (the rest). Note that many students think that function 10 is a decreasing function. Thus, this categorization lends itself to examining definitions and refining the learners' conceptual knowledge.

There are other criteria by which the cards may be sorted/grouped, each highlighting certain properties and distinctions. The activity requires formulating well-defined categories – a non-trivial task, and managing to identify which objects satisfy each category. The need for refining conceptual understanding arises from the specific objects that are chosen for the cards. It calls for addressing mathematical subtleties. For example, functions 1 and 15 seem similar in many ways, including the kind of symmetry they have; yet one is an odd function (1) and the other is not. If the set of cards consist of mathematical problems, then sorting them may lead to better appreciation of the deep structure of the problems, rather than surface features as the content of the problem (E.g., English & Sharry, 1996).

This type of task fosters mathematical fluency. The setting encourages each group to sort the cards by as many different criteria as possible. Thus, the role of the teacher is to help students see these many ways, and classify the cards consistent with the categories. It is not a matter of reaching one "correct" solution.

2. Considering alternatives:

Typically, in this kind of comparison task students are asked to consider a number of possible solutions to a given problem, or a number of mathematical statements describing a concept, from certain viewpoints. For example, various solutions to a given problem or proofs of a mathematical statement may be considered with respect to their appropriateness, correctness, and/or personal preferences. Mathematical statements can be compared also in terms of their validity, equivalence, and defining properties.

Figure 2 is an example of such a task that was given to 12th grade students (Zaslavsky & Shir, 2005).

Consider the following statements. Each one describes a square. Which of the statements would you accept as a definition of a square? Of the statements that you accept as definition, which one do you prefer?   ***Share with your peers your reasoning. Can you reach a group agreement?***
A *Square* is:
(a) a rhombus with a right angle.
(b) a rectangle with four equal sides
(c) a parallelogram with diagonals which are equal, and perpendicular.
(d) a quadrangle with diagonals that are equal, perpendicular, and bisect each other.
(e) a quadrangle in which all sides are equal and all angles are 90º.
(f) a polygon with four equal sides and four equal angles
(g) the locus of points for which the sum of their distances from two given perpendicular lines is a positive constant.
(h) an object that can be constructed (in the Euclidean Plane) as follows: Draw a segment; from each edge erect a perpendicular to the segment, in the same length as the segment (both in the same direction). Connect the other 2 edges of the perpendiculars by a segment. The 4 segments form a quadrangle that is a square.

Figure 2: Task involving consideration and preferences of alternatives (taken from Zaslavsky & Shir, 2005)

This task elicited a lot of discussion and attention to necessary and sufficient conditions of a definition. Statement (e) is the commonly used textbook definition to which students are exposed. The other statements are mathematically equivalent, thus, theoretically each could serve as a definition of a square. However, they differ along other features – minimal vs. non-minimal conditions, structural vs. procedural and degree of hierarchy (Zaslavsky & Shir, 2005). Apparently, the comparison between the given statements elicits students' conceptions of a mathematical definition, as well as their personal preferences.

Mathematics is often regarded as a discipline in which there is always one correct answer for every (solvable) problem (Schoenfeld, 1989) and no room for subjective views. This type of open activity may be helpful in conveying a different view of mathematics, that is to say, a view of mathematics as a more humanistic discipline, in which results are socially constructed or rejected, driven by personal values (Brown, 1982; Kleiner & Avital, 1984). The task calls for personal preferences, not just issues related to correctness.

In the design of such tasks, the statements need not be equivalent. They may draw students' attention to the difference between a definition that forms a necessary and sufficient condition to a necessary but not sufficient of sufficient but not necessary conditions.

3. Compare and contrast:

Typically, in such a comparison task it is customary to present two or more "objects" and ask questions like: In what ways are they similar? In what ways are they different? List as many differences and similarities; Discuss them with your peers. Reflect on what you have learned through this task.

Find as many similarities and differences as you can between the following parallelogram and trapezoid.

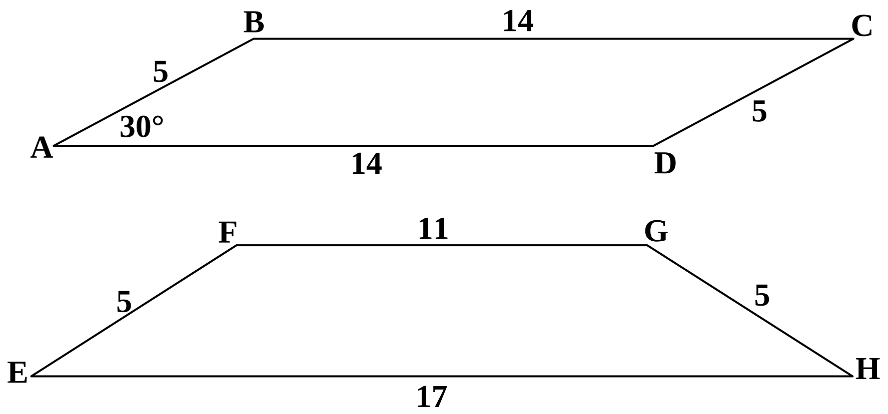

Figure 3: Compare and Contrast Task

Figure 3 is an example of such a task. It requires careful comparison between the two objects. A rather surface similar feature could be that they are both quadrilaterals. They both have at least one pair of opposite sides that are congruent; the parallelogram has two such pairs while the trapezoid has just one pair. Examining the specifics leads to realizing that they have the same perimeter. It may be tempting to conclude that they also have the same area, while in fact, they have different areas. The heights of both can be rather easily calculated; both are rational numbers (2.5 for the parallelogram and 4 for the trapezoid).This is surprising, since a random choice of givens are not likely to imply this (e.g., if we substituted all the sides of 5 unit length by a 6 unit length side, the parallelogram would still have a rational number length height (3), however, the trapezoid's height would be irrational: $\sqrt{27}$). From here it is easy to infer that the area of the parallelogram is smaller than that of the trapezoid. Both figures are symmetrical, however – the parallelogram has point symmetry while the trapezoid has line symmetry.

The design of this task requires objects that have some features in common, yet differ along other features. This task encourages students to examine these objects openly, and while they look for similarities and differences they need to do some calculations that serve a larger purpose (as opposed to a question that asks the students to find the areas of the two figures).

Compare and Contrast tasks can be designed with a larger number of objects of different sorts.

Concluding Remarks

To conclude, let me refer you to Swan's conclusion (this session). The tasks described above have some of the affordances that Swan lists: They expose and build on existing interpretations and understandings (Lloyd & Wilson, 2002); they use cooperative small group work; they are rich open tasks with multiple entry points and many pathways; they value improved comprehension on completion of 'products'. In addition, they all involve a degree of uncertainty as to how to proceed and what the 'solution' is (Zaslavsky, 2005), which could be a driving force and inner motivation for learning.

My experience is that many teachers are reluctant to incorporate such tasks in their classrooms. They are concerned with 'coverage' and loss of control. However, the teachers that were ready to try them out reported on meaningful learning that took place, on students' increased involvement and active participation, and on students' enthusiasm. In addition to the cognitive merits of these kinds of tasks, they have the potential of changing the socio-mathematical classroom norms as well as students' dispositions towards mathematics.

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