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TASK FRAMEWORK/PRINCIPLES

There is no problem of greater importance in the field of task design than that of designing a task that will lay the conceptual groundwork for a new category of mathematical thought. Such a task must provide a basis of sufficient generality to subsume subsequent developments of the concept in question. Bartlett's (1932) research revealed the deleterious effects of introducing new information into an inadequate schema, and Skemp (1987) described the enormity of cognitive effort required to restructure an inadequate existing schema, with the result that such restructuring is rarely accomplished. Schmittau (2003) argued that the practice of developing number from the action of counting creates an inadequate basis for the development of the real number system, since fractions and irrationals do not arise from counting, and cannot therefore, be subsumed into a schema for number predicated on counting. Consequently, the natural numbers function as a generative metonymic for the real numbers, with the result that other instances of the category are not well understood, and many adults do not even consider fractions to be numbers (Skemp, 1987).

V.V. Davydov observed the historic upheavals in science and mathematics occasioned by the need to restructure a schema when confronted by subsequent developments it proved inadequate to subsume, and sought to obviate the need for such upheavals in the pedagogical experience of students. His elementary mathematics curriculum therefore develops the initial schema for number from measurement (rather than counting), from which fractions and irrationals arise as naturally as the counting numbers. From a Vygotskian/Davydovian perspective, the creation of a schema adequate to subsume future developments of the concept of interest is thus the first principle in the design of a task which introduces a new category of mathematical thought. There is, in addition, a second principle which was held by Vygotsky, viz., that for complete comprehension, a concept must be traced from its genesis through its developmental path. Davydov's implementation of this principle is reflected in a series of tasks that require children to make progressively more precise comparisons of quantities, which lead eventually to measurement, from which number is then defined.

EXPONENTIATION

The conceptually deleterious effects of developing number from counting are reflected in the development of operations such as multiplication as repeated addition (which can only occur an integral number of times and hence, does not encompass fractions or irrationals), and are compounded in the development of exponentiation as repeated multiplication (which yields only positive integer exponents) (Schmittau, 2003). Neither of these genetic bases is adequate to subsume fractional, irrational, or negative instances, with the result that these operations are perceived by students as unconnected procedures across the various numerical sub-domains to which they are applied, resulting in instrumental rather than relational understanding of non-positive integer instances (Skemp, 1987). To address this situation for exponentiation, a task was created that not only reflects its historical genesis, but develops an initial schema sufficiently general to subsume real number instances of exponentiation.

THE TASK

At 8:00 Sunday morning a child notices a small plant growing near his house. He decides to measure it and finds that it is 3 cm high. He measures it again on Monday morning at

8:00 and finds it to be 9 cm high. He decides to measure it at the same time on ensuing mornings. Tuesday’s measurement is 27 cm, and Wednesday’s is 81 cm. Assuming that this growth pattern is descriptive of the entire growth history of the plant:

1. How tall was it on the previous Saturday morning at 8:00? Why didn’t he notice it?

2. How tall was it the previous Friday at 8:00 AM? The previous Thursday at 8 AM?

3. If we label Sunday as Day “1”, the first day the child measured the plant, and want to be consistent with our numbering scheme, how should we number the following days: Saturday? Friday? Thursday? If we denote the height of the plant on Sunday at 8:00 AM as 3^1 cm, how would we express the heights on the other days?

4. How tall was the plant at 8:00 the previous Saturday night? At 8:00 Sunday night? At 4:00 p.m. on Saturday? (Did you happen to find the height at another day or time?)

5. When will the plant be 46.765 cm tall?

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Commentary: This task can be used after introducing the symbolic representation (2^3 , 3^4, x^n), for positive integer exponents. It is not designed to reflect botanical realism, but rather to provide for the two important Vygotskian/Davydovian principles described above. By “front loading” the initial development of the concept of exponentiation, and working from its genesis, which historically entailed movement between arithmetic and geometric sequences (appearing here as the “Day” and “Height” axes, respectively), through the development of the exponential *function* $y = 3^x$, and solving for x at various intervals, not only positive integer, but zero, negative, and fractional exponents are developed. Because of the continuous nature of plant growth, heights involving irrational exponents can also readily be seen to be both necessary and plausible.

Part 5 generates a need for the development of logarithms, which can be further motivated for students by observations such as: “This problem could not have solved for most heights by the use of our methods. We were asked to solve $3^x = 46.765$, or in alternate symbolic form, $x = \log_3 46.765$. Tables for logarithms to the base ten have been worked out for the decimal system, and for the natural logarithmic system, which will be studied later. Both are calculator accessible.”

Note: No claim is made that Davydov or his colleagues would have developed or employed precisely this task, but merely that it meets the two criteria described above, to which Davydov, following Vygotsky, subscribed.

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BIBLIOGRAPHY

Bartlett, F. (1932). *Remembering.* Cambridge, UK: University Press.

Davydov, V.V. (1990). Types of generalization in instruction: Logical and psychological problems in the structuring of school curricula. Reston, VA: National Council of Teachers of Mathematics

Davydov, V.V., Gorbov, S.F., Mikulina, G.G., & Saveleva, O.V. (1999). *Mathematics-Grade 1* (J. Schmittau, Ed,), Binghamton, NY: State University of New York . (Translated from the Russian by O. Sviridova & S. Gozenpud.)

Lakoff, G. (1987). *Women, fire, and dangerous things: What categories reveal about the mind.* Chicago, IL: University of Chicago Press.

Schmittau, J. (2003). Cultural historical theory and mathematics education. In A. Kozulin, B. Gindis, S. Miller, & V. Ageyev (Eds.), *Vygotsky's educational theory in cultural context*, pp.225-245. Cambridge, UK: Cambridge University Press.

Schmittau, J. (1994). Scientific concepts and pedagogical mediation: A comparative analysis of category structure in Russian and U.S. students. Paper presented at the Annual Meeting of the American Educational Research Association, New Orleans, April, 1994.

Skemp, R. (1987). *The psychology of learning mathematics.*

Vygotsky, L. S. (1986). *Thought and language.* Boston, MA: MIT Press.