DESIGNER CONCERNS VERSUS STUDENT WORK: THE CASE OF *IMPROVING GRADES*

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The design of mathematical tasks is a complex process. In our proposed presentation, we will identify and examine some components of the design process of investigative activities.

In order to analyze this complex process, we chose an exemplary activity in algebra termed *Improving Grades*. We plan to present here some considerations related to designing an investigative activity in general. In addition, we will discuss how these considerations apply to our particular activity, analyze student work on the activity and finally, compare the designers' initial intentions with aspects of students' work on the activity.

DESIGN CONSIDERATIONS

Mathematical content. The mathematical agenda (e.g., concepts, algorithms, properties, and definitions) is one of the main considerations that determine the choice of a particular activity. Activites based on other considerations (such as intended cognitive processes, technological instrumentation or choice of representation) but lack a clear mathematical agenda might not contribute clearly to the aim of learning mathematics as a body of knowledge, and are prone to criticism by many mathematics educators, mathematicians, and the public at large (Wu, 1994).

Context of inquiry. A context-based approach to learning mathematics employs real-life or mathematical problem-situations in order to provide meaning to concepts and operations (Gravemeijer & Doorman, 1999). A context-based approach has both immediate and more general advantages.

- It facilitates learning processes by providing real or concrete meaning to an otherwise abstract concept or algorithm (Heid et al., 1995).
- It provides points of reference that students can review at a more advanced stage of learning, when work is performed at a more abstract level.
- It raises student motivation and willingness to become engaged in the learning activity.

However, in spite of its potential to promote a deeper understanding of mathematical concepts, a context-based approach is prone to cause some cognitive difficulties as well. Thus, for different students, a "familiar" context tends to be "familiar" in different ways. As a result, students might understand a mathematical concept in unintended ways, or follow an "unproductive" path of solution (Sutherland, 2004).

Level of openness. The decision regarding the level of openness of an activity is based on a constant state of tension between unstructured and open versus structured tasks. The solution of an open problem does not require a specific method, a certain representation, or an implicitly given sequence of steps, whereas structured problems pose specific requests with regard to the variables mentioned above. The open approach is intended to develop problem-solving skills, develop creative mathematical thinking, provide opportunities for students to actually experience investigations, and achieve a meaningful construction of knowledge.

The structured approach enables students to pursue a more predictable and planned learning trajectory in specific domains of mathematical content and to better understand the processes of problem solving. However, a structured sequence of tasks imposes a unique and clearly defined learning trajectory that does not necessarily meet the needs or preferences of all students.

Representations. Use of various representations is recommended for several reasons (Ainsworth, Bibby, & Wood, 1998): (1) they express different aspects of a given situation more clearly (Goldin, 2002); hence, the information gained from combining representations will be greater than what can be gained from a single one; (2) different representations constrain each other, so that the space of permissible operators becomes smaller; (3) when required to relate multiple representations to each other, the learner has to engage in activities that promote understanding.

Sequencing of tasks within an activity. Most mathematical investigations follow an inductive path based on a transition from the investigation of particular cases to pattern generalizations, then to justifying the evolving pattern and later on, to its implementation in additional cases (Friedlander, Hershkowitz, & Arcavi, 1989).

In the next section, we will examine the five design considerations described above in the context of a specific investigative activity.

ANALYZING THE ACTIVITY

In the *Improving Grades* activity (adapted from Arcavi, 2002), students are required to compare the following three "schemes" for improving grades on a "difficult test":

- Scheme A: Add 10 points to the original grade
- Scheme B: Add one fourth of the original grade
- Scheme C: Average the original grade and 100.

Mathematical content. The activity was designed for an 8th grade algebra course. The content-related purpose of the activity is (1) to enable students to create and use numerical, graphical, and symbolic representations, and (2) to make the process of learning algebra in general and algebraic modeling in particular, meaningful and effective (Friedlander & Tabach, 2001).

Context of inquiry. Grades and grade improvement are rooted in students' school life. As a result, the context has the potential to raise student motivation, to be a source of a variety of authentic questions, and to enable students to monitor and reflect on the obtained results.

Level of openness. The following requirements in the *Improving Grades* activity appeared to differ from most traditional algebra tasks with regard to the activity's level of openness.

- The questions are posed in a natural "spoken" (as opposed to "school") language that does not channel students' work in a certain direction.
- Students are asked to compare several processes of variation.
- Students are asked to construct various representations at an initial phase, before starting to investigate the problem.
- The questions do not specify, recommend, or require a particular representation that should be used in the solution.
- Students are asked to specify what representation has been used to solve each question.
- The solution of each question enables students to employ a variety of representations.

On the other hand, the activity leads the student to solve the problem through a carefully structured sequence of questions that gradually move from local, numerical instances, to more general situations based on equalities between two unknown quantities and finally to investigation of variation patterns.

Representations. Initially, the grading schemes are presented verbally. Before starting their investigation, the students are required to construct some exemplary data (pairs of original and improved grades), and to represent them numerically (fill in a table of numbers), symbolically (construct a symbolic expression for each scheme), and graphically (draw the corresponding graphs in the same coordinate system). The requirement to produce a variety of representations *before* expressing a need to employ them is intended (1) to raise awareness of the possibility of employing different representations in different situations, and (2) to facilitate the choice of a particular representation, without the need to invest additional effort in its construction or alternatively, to avoid its use.

Sequencing of tasks within the activity. The activity poses a variety of questions, aimed at leading students through their work. For each question, students are asked to provide explanations and to show their work. Additionally, at the end of each question, the students are asked to circle the appropriate term for the representation used to solve the particular question.

The activity tasks were sequenced according to the following three criteria: (1) task complexity, (2) grading scheme complexity, and (3) cognitive complexity.

1. *Task complexity*. Within a certain task, the students were required to relate to one, two, or three grading schemes. Accordingly, the tasks were grouped into two categories:

   Getting acquainted: at the beginning of their investigation, students are asked to relate to each scheme separately -- first by finding improved grades for given original grades (i.e., "direct questions" based on an original grade → improved grade direction), and then by finding the corresponding original grades for given improved grades (i.e., "reversed questions" based on an original grade ← improved grade direction). As designers, we anticipated the use of numerical representations for direct questions, and the use of a symbolic algebraic representation when working in the reversed direction.

   Comparing schemes: for the next three questions, students are asked to solve two cases of grade equality (i.e., finding the original grade that produces the same improved grade for two schemes), and one case of grade inequality (i.e., finding the domain of the original grades that produce the highest improved grade for a given scheme, as compared to the other two). We assumed that the grade equality questions require a more local approach, as compared with the global considerations required in the more complex situation based on an inequality. As a result, we anticipated the use of graphical or symbolic representations for the equality tasks, and the use of graphical representations for inequality tasks.

2. *Grading scheme complexity*. We arranged the presentation of Schemes A, B, and C in the activity (x + 10, 1.25x or $x+\frac{x}{4}$ and $\frac{x+100}{2}$, respectively) in an increasing order of complexity of their underlying rules.

3. *Cognitive complexity*. The sequence of tasks within the activity was also determined by assuming that local direct, and then reversed questions are cognitively less demanding than the global grade equality followed by the inequality questions.

In the next two sections, we will first analyze some aspects of the experimental students' work on the activity, and then examine these students' outcomes in view of our initial design considerations.

ANALYSIS OF STUDENT WORK

Sixty-one students from three 8th Grade algebra classes worked on the activity during a ninety-minute class period. We examined student data on solutions of mathematical tasks, and on questions, which in each case, specifically requested the students to state their choice of representation.

Table 1 presents examples of students' solutions on two reversed questions (original grade ← improved grade), one case of equality (the same improved grade according to two schemes), and one case of inequality (the grade according to one scheme is larger than the corresponding grades according to the other two schemes). To illustrate the variety of solutions employed by the students, we selected examples of numerical, symbolic, and graphical solutions of each item.

Table 1. Examples of students' use of various representations for the same question*.

<table>
<tr><th>Question</th><th>Solution</th><th>Represent.</th></tr>
<tr><td rowspan="3">Scheme B.
original grade ⟵ improved grade

Ron's new grade is 72.
What was his original grade?</td><td>$\frac{72}{5} \bullet 4 = 57.6$</td><td>numerical</td></tr>
<tr><td>$x + \frac{x}{4} = 72$ $4x + x = 288$
$x=57.6$</td><td>symbolic</td></tr>
<tr><td>I looked at the Y axis 72, and I saw that the original grade is 57.</td><td>graphical</td></tr>
<tr><td rowspan="3">Scheme C:
original grade ⟵ improved grade

Rachel's new grade is 81.
What was her original grade?</td><td>81•2 – 100 = 62</td><td>numerical</td></tr>
<tr><td>$\frac{x+100}{2} = 81$ $x + 100 = 162$
$x = 62$</td><td>symbolic</td></tr>
<tr><td>I looked at the graph, and I found that the original grade is around 60.</td><td>graphical</td></tr>
<tr><td rowspan="3">Grade equality:
Scheme A = Scheme B

Raz told his friends: "I don't care if the class chooses Scheme A or Scheme B".

What was Raz's original grade, and what is his new grade? Explain.</td><td>40 and 10 are 50, and a quarter of 40 is 10, plus 40 is 50.</td><td>numerical</td></tr>
<tr><td>$x + 10 = x + \frac{1}{4}x$ $4x + 40 = 4x + x$
$x=40$</td><td>symbolic</td></tr>
<tr><td>It is clear from the graph that if your original grade is 40, then both in Schemes *A* and *B* your improved grade will be 50.</td><td>graphical</td></tr>
<tr><td rowspan="3">Grade inequality:
Scheme C > Schemes A or B

Alex told the others: "I would prefer that the class chooses Scheme C".

What was Alex's original grade? Explain.</td><td>Alex's original grade is less than 67, hence:
<table><tr><th></th><th>A</th><th>B</th><th>C</th></tr><tr><td>66</td><td>67</td><td>82.5</td><td>83</td></tr><tr><td>67</td><td>77</td><td>83.75</td><td>83.5</td></tr><tr><td>68</td><td>78</td><td>85</td><td>84</td></tr></table></td><td>numerical</td></tr>
<tr><td>$x + 10 < \frac{x+100}{2}$ and
$x + \frac{1}{4}x < \frac{x+100}{2}$</td><td>symbolic</td></tr>
<tr><td>Alex's [original] grade is less than 65 (approximately). I looked at the graph and saw that the part for which Alex's graph [scheme C] is the highest is up to 65 (approximately).</td><td>graphical</td></tr>
</table>

* ▢ Representations that received a considerably higher degree of preference, as compared to others for each question.

Table 2 presents the frequencies of the students' choice of representations. The emerging solution patterns were as follows:

Table 2. Students' use of representations by question (in percent, N = 61)*.

Task	Scheme	Numerical represent.	Symbolic represent.	Graphical represent.	Mixed represent.
original grade → improved grade	Scheme A	***88***	10	2	-
	Scheme B	***87***	5	-	8
	Scheme C	***82***	10	5	1
original grade ← improved grade	Scheme A	***88***	10	2	-
	Scheme B	***48***	***41***	7	4
	Scheme C	***52***	***30***	10	2
grade equality	A = B	16	***64***	9	11
	B = C	5	***72***	13	4
grade inequality	C > A and B	12	***36***	***35***	7

* [shaded box] Representations that received a considerably higher degree of preference than others for each question.

- The direct questions (original grade → improved grade) attracted numerical representations for more than 80 percent of the students.
- Most of the reversed direction questions (original grade ← improved grade) received mainly numerical and symbolic solutions. In these cases, about half of the students still used a numerical representation, but many others (about a third) preferred to solve these questions using symbolic equations. As an exception to this pattern, the solutions of the reversed direction questions for Scheme A attracted almost exclusively numerical solutions, owing to the simplicity of Scheme A (add/subtract 10 points). Therefore, we can still conclude that reversed questions often create the need for symbolic thinking.
- The preferred solution method for grade equality questions was the use of symbolic equations. The first equality question had an integer solution and attracted slightly more numerical solutions, as compared with the second equality question, which had a non-integer solution.
- The inequality question required finding a domain satisfying an inequality; it attracted an almost even division between graphic and symbolical representations (with about one

third of the students for each). Thus, questions that require a more global view of a situation seemed to encourage the use of graphical solutions by more students.

Note that for each question, the use of one or at most, two preferred representations did not prevent some students to use other representations.

Another aspect of variability can be detected if we consider the number of representations employed by individual students. Only ten percent of the students used one single representation throughout the activity. In contrast, 39, 43, and 8 percent used two, three, and four representations, respectively, in their work. In our view, these percentages indicate that most students were aware of the advantages of using different representations for different questions and were able to employ them in their solution.

As mentioned above, the activity tasks were sequenced according to task complexity (relating to one, two, or three schemes), grading scheme complexity (Scheme A, Scheme B, and Scheme C in increasing order), and cognitive complexity (first local direct and reversed questions, and then more global grade equality and inequality questions). Table 3 presents the students' average scores on the rubrics of each of these criteria.

Table 3. Student average scores on three criteria (in percent, N = 61).

Criterion	Rubrics	Average scores
Task complexity	One scheme	92
	Two schemes	79
	Three schemes	72
Scheme complexity	Scheme A	94
	Scheme B	91
	Scheme C	92
Cognitive complexity	Direct	98
	Reversed	85
	Equality	79
	Inequality	72

The results clearly indicate a descending level of achievement on the rubrics of task complexity and cognitive complexity, and an almost uniformly high level of achievement by grading scheme complexity.

DESIGN INTENTIONS VERSUS ASPECTS OF STUDENT WORK

In this section, we will discuss how our five original design considerations manifested themselves at the stage of the activity's class implementation.

Mathematical content. Throughout their work on the activity, the students substituted numbers in algebraic expressions, solved equations, as well as drew and analyzed graphs. Moreover, they applied these skills to a variety of tasks in a mathematically sound and selective manner.

Context of inquiry. We assume that the students' familiarity with the context of grading and with the activity's natural "spoken" (as opposed to "school") language helped them understand the mathematical tasks at hand. In our experimental setting, we could not establish a direct connection between activity context and student achievement. However, the fact that the activity was based on a relevant and meaningful context was likely to contribute to the students' relatively high level of performance.

Level of openness. We attribute the students' variety of solving strategies (Tables 1 and 2) to the activity's relatively high level of openness established in the design process (see the discussion on openness in the task analysis section above).

Representations. The findings indicated some patterns of student choice of representations (see Table 2). The gradual and continuous transfer from preferred numerical, to symbolic and graphic solutions (see the left-to-right shift of the shaded cells in Table 2) shows a correspondence between the designers' intended, and the students' actual learning trajectory, as expressed in preferred representations.

Students' work on the *Improving Grades* activity showed that direct questions usually encouraged the use of a numerical approach. In contrast, reversed questions and equality questions encouraged a symbolic or a graphical approach, whereas situations based on an inequality made the use of graphs more convenient. The choice of a symbolic or a graphical solution may also depend on students' personal preferences or on the extent of their mastery of the involved skills. Thus, most of the eighth graders who took a second-year algebra course preferred to work symbolically rather than graphically whenever a numerical approach was inconvenient. A similar study of younger, first-year-algebra students (Friedlander & Tabach, 2001) showed a stronger preference to use a graphical representation – probably because of their lack of experience with symbolical manipulations.

Sequencing of tasks within the activity. The average achievement scores on the criteria of task complexity and cognitive complexity (Table 3) indicate that the students' work followed the sequence established by the designers. In fact, we could not distinguish any significant differences according to grading scheme complexity, owing to an almost uniformly high level of performance on this criterion.

CONCLUSIONS

In this presentation, we sought to identify and examine five components of the design process of an investigative activity: mathematical content, context of inquiry, level of openness, representations, and task sequencing. We analyzed these components at various stages of the design process – initial planning, activity design, and activity implementation. We showed that each stage plays an important role in the design process. At the stage of initial planning, the designers establish their mathematical and pedagogical agenda in general. Analysis of the designed activity is needed to ensure the presence of the established agenda in the activity. The findings collected at the stage of classroom implementation enable one to compare the designers' hypothetical and the students' actual learning trajectory (Simon & Tzur, 2004).

Finally, we would like to make two cautionary remarks:

- Within the framework of an intended, enacted, and attained curriculum (Clements, 2002), we focused on the initial stage of establishing the intended curriculum and made an initial estimation with regard to the other two aspects. Accumulating research indicates considerable differences among these three phases in curriculum development.
- We analyzed here the design process of one prototypical but isolated activity. The design of a complete learning sequence or curriculum includes an even larger set of considerations and stages of development.

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