EXPLORING FUNCTIONAL RELATIONSHIPS TO FOSTER ALGEBRAIC THINKING IN GRADE 8

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Abstract. *This article aims to identify possible contributions of a teaching unit conceived to promote grade 8 students' algebraic thinking. The unit was designed within a research project and is based on exploratory and investigative tasks about functional relationships. Using a qualitative and interpretative methodology, data collection included participant observation of classes by the teacher, registered in her journal, and gathering written records from students. Results indicate that working in the proposed tasks and combining group work and classroom discussions, promoted the development of meaning for the algebraic language. Working within this unit in this way also encouraged the students to widen the number of strategies that they used in exploring situations involving relations between variables, to reason in a more general way, and to express their generalizations using a formal language.*

INTRODUCTION

In Portugal, algebra is an important topic of the grade 8 mathematics curriculum, addressing the use of symbols with different purposes and in different situations such as representing functional relationships, solving problems, 1st and 2nd degree numerical equations and literal equations, and generalizing and proving numerical properties (ME, 1991). The students of this grade had former experiences with the algebraic language, particularly in the study of 1st degree equations (at grade 7). However, the use of algebraic language and the construction of the concept of variable usually create significant difficulties for many students. This paper is drawn from a larger research that studies how solving exploratory and investigative tasks, involving functional relationships, may contribute to the development of their algebraic thinking (Matos, 2007). Our main goal is to identify the main contributions of a teaching unit based on this curriculum strategy in students' learning.

ALGEBRAIC THINKING

Kaput (1999) argues that algebraic thinking appears when – through the processes of conjecturing and arguing – one establishes generalizations about data and mathematical relationships expressed in an increasingly formal language. This generalization process may occur from arithmetic, geometric, and mathematical modelling situations. The author identifies five faces of algebraic thinking, intimately related: (i) generalizing and formalizing patterns and constraints; (ii) manipulating formalisms; (iii) studying abstract structures; (iv) studying functions, relations, and joint variation; and (v) using multiple languages in mathematical modelling and control of phenomena. In this way, Kaput stresses the need for a wider way of looking at algebra teaching and learning. This idea is also underlined by the NCTM (2000) that indicates that middle school students must learn algebra, both as a set of concepts and skills related to the representation of quantitative relationships and as a style of thinking that enables the formalization of patterns and generalizations.

Although algebra is not just a language, it is true that some of its power comes from the use of symbols that allows expressing mathematical ideas in a short and rigorous way (Sfard & Linchevski, 1994). Symbols also allow keeping a distance from the semantic elements they represent becoming powerful tools for problem solving (Rojano, 1996). Many authors point that symbols can be used to represent different mathematical aspects and can be interpreted by the students in different ways (Küchemann, 1981; Usiskin, 1988). Namely, symbols can be used in the representation of unknown numbers, in the expression of generalizations, or as variables. Some authors also identify several difficulties students reveal when they deal with the algebraic language. Booth (1984) indicates three main areas: (i) interpreting letters; (ii) formalizing the methods used; and (iii) understanding notations and conventions. In fact, the multiple uses of the algebraic symbols are a source of potential in algebra but also a source of conflicts and difficulties for the students.

Driscoll (1999) and the NCTM (2000) point out that the exploration of patterns is an activity that contributes towards the development of algebraic thinking and that must be promoted since the first years of school. We also note that the study of functions includes the need to understand the way how two variables are related, based in the identification of regularities. Concerning this issue, Smith (2003) identifies two distinct ways of analysing a function: (i) understanding the relationship between each value of the variable x and the associated value of y, which may enable to write an algebraic expression that represents it; and (ii) analysing the way how the variation of the values of a variable produces variation in the values of the other, that is, analysing covariation of x and y.

THE TEACHING UNIT

This teaching unit involved the study of several topics of the mathematics curriculum – numerical sequences, functions, and 1st degree equations. Aiming to promote the development of algebraic thinking, based on exploring functional relationships, its specific objectives were the development of the students' ability to: (i) identify and describe patterns and regularities in situations involving variation and to formulate generalizations; (ii) to represent and analyse functional relationships through tables, graphs and algebraic expressions; and (iii) to ascribe meaning to algebraic expressions and to deal in an efficient way with the algebraic language. Students must use letters in different contexts and with different purposes – as instruments for generalizing, as variables in functions and as unknowns in solving problems and equations.

The unit was carried out in 16 classes (90 minutes each). It included many types of learning experiences. The first part of the unit included exploratory and investigative tasks (Ponte, Brocardo, & Oliveira, 2003) in the beginning of the study of each topic, as a mean to foster the construction of new concepts. In the tasks about numerical sequences, students had to explore patterns in several sequences, with or without geometrical representations and with different levels of difficulty. All of these tasks created opportunities for pattern generalizing, which could be expressed in the beginning in natural language but should progressively be expressed in a more formal way, using algebraic language. In the third task, the students worked with numerical sequences graphically represented. In this first part of the unit, letters were mainly used as generalized numbers and as unknowns in simple 1st degree equations.

In the second part of the unit, the study of functions was introduced by two tasks involving relationships between variables. They represented an extension from the discrete case to the continuous case. The first task involved a direct proportion. Both tasks explored different ways of representing functional relationships, moving from one kind of representation to another and addressing the potential of each one. The third task had four distance-time graphs that students should interpret to design a situation adjusted to the information given. Although the letter is used both as a generalized number and as an unknown, in this part of the unit the focus was on its use as a variable and on the notion of joint variation.

The last two tasks continued the study of equations that the students begun at grade 7 and re-visited in previous topics, solving new kinds of problems and equations with denominators. Literal equations appeared through the investigative activity that the students developed as a result of generalizing a relationship between more than two variables. After this initial moment, supported by the context of the situation, the students got involved in solving these equations in order to one of the variables. In this phase, letters were mostly used as unknowns and as generalized numbers. All of the tasks allowed the students to use different strategies and to draw their own paths of exploration. This approach stimulates their active participation giving them multiple entry points, adequate to their ability levels.

Identifying patterns and regularities, representing, generalizing and particularizing are mathematical reasoning processes that play an important role in exploring functional relationships. Solving investigative tasks may allow the students to learn through the mobilization of their own cognitive and affective resources, as they follow their own goals (Ponte, 2006). This approach, on the beginning of each topic, complemented by a small set of instructions, seeks to empower the students' intuition, as they involve themselves on the exploration of the tasks, initially in an informal way. The unit includes also some exercises and problems from their textbook.

These classes included individual work and work in pairs and in small groups. At the end of each task or group of tasks a discussion involving all class took place. In these discussions, the students shared orally their strategies with their colleagues. While they were solving the tasks, they made written records of their work which allowed them to organize their reasoning and supported their performances in the general discussions. In two of the tasks these records where analyzed by the teacher which addressed each group some new questions. These questions stimulated the students to go beyond their first explorations, in the next class.

METHODOLOGY

This article aims to identify possible contributions of a teaching unit focused on exploratory and investigative tasks about functional relationships on students' learning. We used a qualitative methodology, descriptive and interpretative (Bogdan & Biklen, 1994). The unit was taught to a grade 8 class from a school of the Lisbon suburban area. The first author of this article was the classroom teacher and she performed simultaneously the role of teacher and researcher. This was an investigation about her own professional practice. The class had 27 students (aged 13 to 16). Reflecting the increasing immigration rate observed nowadays in Portugal, 10 of these students were from former Portuguese colonies – Angola, Cape Verde, S. Tomé and Príncipe, Guinea and Brazil – or from Eastern European countries, such as Romania. The students in this class had low achievement in almost all the subjects and several of them did not succeed at least one year. It existed, however, a good relationship between them and their teachers.

Data for this paper is taken from the teacher's journal, including her daily observations and complemented by data from audio recording of all classes and written documents produced by the students, collected and organized. The data analysis was descriptive and interpretative and was done in an inductive and exploratory way (Bogdan & Biklen, 1994). This analysis considers the main episodes occurred in the classroom during the unit, regarding the strategies that the students used to explore the tasks and the main difficulties they revealed while they were solving them.

RESULTS AND DISCUSSION

Numerical sequences. The students were puzzled with the first exploratory task. I (the first author) indicated them that they should work in pairs, making notes from their conclusions. Since this was a new topic for them, they expressed difficulties in interpreting what was sought in some questions. When I noticed this, I suggested them to look attentively at the figures and share ideas with their colleagues. After this initial moment, the students drew their attention back to the task and detected the first patterns, which gave them more encouragement to continue. The first strategies that they used were very intuitive. Most of them analyzed the covariation between the variables to determine the number of points on each figure, adding two units to the previous figure. Other students represented all or just some of the figures and counted directly the points they saw. Filipa and Carla chose a different strategy, based on the geometrical decomposition of the figure in two parts (Figure 1). The first one had as many points as the place it had in the sequence and the other part had one point more:

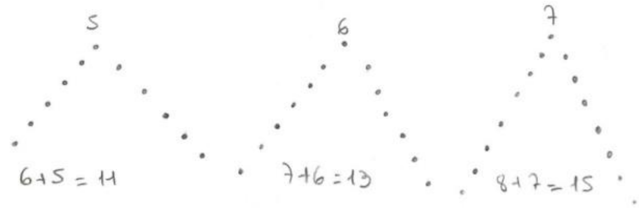

Figure 1 – Filipa and Carla – Task 1 – Numerical sequences

When the students had to describe a distant figure and a general rule of formation for the sequence, they used new strategies, considering also the correspondence relationship between variables. This relationship was expressed on two different ways, but was based on the geometrical decomposition of the figure in three smaller parts: two main branches with the same number of points as the order of the figure and the point situated at the vertex. Adding each number of points, they obtained 100+100+1 or 2 x 100 +1, which totalized 201 points. As they had done initially, Filipa and Carla added just two addends, because they divided the figure in just two parts as we already saw. The students explained that the figure had 201 points, calculating: 101+100=201. Supported by the geometrical representation, several pairs could actually imagine the 100^{th} figure, opening the way to generalizing. However, this was also a point where some students showed some difficulties. Jacinto and Carolina tried to use their knowledge about covariation but they thought it was necessary to add 3 points instead of 2. In this way, they could not obtain the correct number of points.

Concerning generalization there were also distinct answers. It was quite apparent that the way students solved the first questions had a strong influence in how they described a general rule. Some, such as Erica and Miguel, made general statements, directly related to the analysis of covariation: "We always add two points to the existing number." Other students based their generalization in the relation of correspondence that they had previously detected. Therefore, students that in the 100^{th} figure had multiplied the number 100 by 2 and added 1 answered as Pedro and Catarina: "It is twice a number plus the higher point". Filipa and Carla, also following their initial reasoning stated: "When we have any number, we add one point more and then, with the result of the sum, we add that number again." In this initial phase of the teaching unit some of the students were not able to generalize and to describe generalizations in an autonomous way.

The students were already familiar with this kind of classroom organization. They knew that, during the general discussions, sharing their strategies would be valued, especially when they could add something new to what was being said. In the discussion of the first task, when Ricardo went to the board, he explained:

Ricardo: So, we take this one [the figure's order], we multiply it by two and we add one.
Teacher: But was that the first strategy you used?
Ricardo: No, we saw that we had to add two points.

At this point, the student recognized that his group started by observing the way how the number of points varied from one figure to the next. The strategy of using covariation was very immediate for the most of the students. However, this task led the students to go beyond this type of reasoning, looking for relations among different variables and describing them in natural language. Although they had already contact with the algebraic language, none of the groups used it in this task as a tool to express the general rule they had found.

During this general discussion emerged the possibility of representing that information in a more succinct way, using the algebraic language. The students suggested the use of the letter *x*, as they had already done in the previous year to represent unknown values. Taking this idea, I asked the students if they could build a formula to represent the number of points of a general figure. Joaquim wrote immediately on his worksheet the expression $2 \times x + 1$ and called me, saying, in an enthusiastic way: "Teacher, this is our rule, isn't it?" Before I confirmed this answer, I tried to know what the other students had in mind. I saw that a few other students had written the same formula. However, most of the students could not write suitable expressions. Influenced by the reasoning she had made before about the 100^{th} figure (to multiply the number of the figure by two and add one), Erica suggested: $x \times 2 = x + 1$. After this, Ricardo suggested $x \times 2+1$ completing his colleague's answer.

Filipa and Carla obtained different algebraic expressions, based on the previous decomposition of the figures: they made: $x + 1$ points to one side and x points to the other. Adding these two expressions, other student suggested that $x + x$ was equal to $2x$, which lead us again to the formula, so we got the expression $2x + 1$. The students continued sharing their own algebraic expressions. Erica for instance, suggested again: $x.2 + 1$. Miguel contributed with other possible expressions: $1 + 2x$ e $1 + x.2$. This discussion about equivalent expressions was an important part of the class. At the end, regarding the expression $2x + 1$, Jacinto remembered something he seemed to have learnt before: "Oh, we have to isolate…!" At that moment I saw that the student was referring to the procedures he was taught to solve equations, without noticing that this was only an algebraic expression. In this case, the aim was to generalize and not to determine the value of an unknown. This thinking was corrected by Joana who said: "No, that is only when we have an equal sign …" This dialogue was followed by our first reflection about the difference between an equation and an algebraic expression.

The next task allowed students to contact with numerical sequences that were not related to geometric representations. The most part of the students started again to find regularities among consecutive terms. The use of tables to represent the sequences was important because it turned more visible the correspondence between the variables, favouring generalizations. The students had also the opportunity to work with the graphical representation of the sequences and to solve small problems in which they had to find the order of some of the terms. Many students showed they were able to invert their initial operations. This was a first step in formalizing the reasoning, that later became quite important, especially in solving equations.

Linear functions. The study of functions began with the analysis of two contextualized situations that could be modelled by linear functions. Both of them referred to shopping with or without discounts. The students started their work immediately building tables with some concrete cases. This situation was important because it created the opportunity to discuss about the values that made sense to use in that context. It was also interesting to verify that the students used the same strategies they had already developed in the study of sequences.

One of the main difficulties they revealed was to analyze and describe the variation of the variables. After this initial moment, the students started to analyze covariation, considering equal increments on the independent variable: "They vary with each other. If you increase one litre you increase also 1,1 € in the price you pay" (Jacinto and Florbela). Other students analyzed covariation and also the correspondence between variables: "For each litre more, we add 1,1 € or we multiply the number of litres we buy for 1,1 €.

At this phase, the students did not show difficulties in generalizing. When they tried to express it using a more formal language, they started to use the same symbols they used in the study of numerical sequences. In this way, the work of some students in the continuous case revealed the influence of their previous work in the discrete case.

Numerical equations. The different experiences the students lived before in algebra was one the aspects that was more visible in this class, especially concerning equations. Only some of the students that had studied this topic in grade 7 could effectively solve a 1st degree equation. The rest of the students, on the contrary, showed they had never learnt how to do it, because they had almost no interest about mathematics or because they had never studied them at their previous schools in the foreign countries.

Working with sequences and functions became an opportunity to use the algebraic language as a tool for generalizing and sharing meanings. The study of these topics generated the opportunity to solve simple equations, which was important to create a common understanding among students, allowing them to continue to the approach of more complex algebraic concepts. In the first general discussion, the study of the sequence with general term $3n + 5$ raised the following dialogue:

> **Teacher:** So, which was the order in which 300 was placed?
> **Erica:** Teacher, 3 x 100…
> **Teacher:** Ok, but does that give 300?
> **Joaquim:** No, that is just with $3n$.
> **Teacher:** Oh, but I can't change the rule like that because we would be working with another sequence, different from this one. We just need to know which is the n that makes this expression yield 300.
> **Sofia:** 300 – 5? I don't know. [Students talk with each other.]
> **Erica:** So, we make $3n = 300 – 5$.

Immediately after this episode, some students did not follow the reasoning proposed by Erica, and went on thinking on their own strategies. Pedro claimed with enthusiasm: "3 x 98 + 5 = 299; 3 x 99 + 5 = 302. It will not pass on 300!" This discussion continued with the contributions of Isabel, who finished solving the equation in the board, according to her previous knowledge. The way how the discussion was developed allowed the confrontation between Erica's idea, the formal resolution proposed by Isabel and the intuitive process used by Pedro to see if 300 was a term of the sequence and the advantages of all the processes.

Literal equations. The last task was one of the most challenging for me as a teacher in this unit. In the beginning I felt difficulties in dealing with the many paths students could choose. One of the situations involved studying the length of a wall built with yellow bricks. That length varied according to the number of bricks that the students could add to the wall in one of two directions: horizontal or vertical. Before the existence of three variables instead of only two, most part of the students chose to consider blocks of two or more bricks that they could repeat in order to build the wall. This strategy allowed them to know that the total length was multiple from the length of each block. The first written records from Marisa and Helena showed this reasoning (Figure 2).

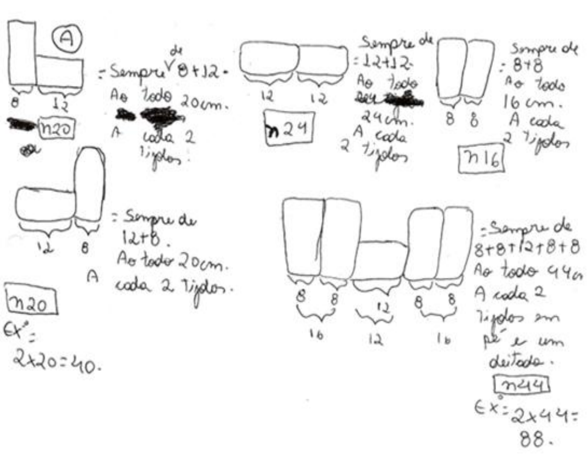

Figure 2 – Marisa and Helena – Task 8

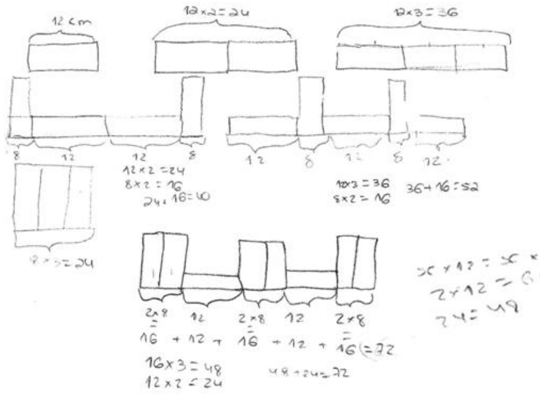

Figure 3 – Sofia and Laura – Task 8

The analysis of walls built with two kinds of bricks brought the need, to some groups, to distinguish the length of the wall that was generated by each of them. Firstly, the students represented both of the lengths using the same letter, which led them to a situation of ambiguity. That was what happened to Sofia and Laura (Figure 3). Their initial reasoning had in consideration different possibilities. However, we can see that the 72 cm came from the bricks in the horizontal direction and the 48 cm came from the bricks in the vertical direction. Although they had used *x* and *y* as the number of bricks in the horizontal and the number of bricks in the vertical the students used the same letter "C" to describe both of the lengths and formulated the following equation to which they could not find any meaning: $x \times 12 = x \times 8$. As they were blocked, they asked for my intervention. I suggested them to think again about the meaning of all the expressions they had used. After this they tried to explain those meanings in their worksheet and, considering C as the total length of the wall, they could write the literal equation $C = 12x + 8y$. This was the first generalization the students made with three different variables represented by three different letters. Other groups in the class reached similar conclusions, through the exploration of other possible walls. As the students explored this situation, first in concrete cases and then as a generalization, the meaning of those expressions became clear for them. As a teacher, the initial difficulty in dealing with so many paths of exploration gave place to the joy of seeing the different types of reasoning the students were able to build and the strategies they could share in the classroom discussion. Analyzing their written records before the general discussion allowed me to be better prepared for that moment.

CONCLUSION

The work in this teaching unit did not represent a reduction of algebra to just a formal language, although this aspect clearly maintained an important role. The situations proposed were based on a wider conception of algebra, involving the understanding of patterns and relationships through the exploration of sequences and functions. The initial part of the unit, with the students' autonomous work and the general discussions, was essential to activate the intuitive resources of students and the knowledge and difficulties they developed in former school years. While sharing their own ideas, the students could clarify their doubts about the meaning of some algebraic expressions and become aware of the advantages that the use of this language could have for expressing generalizations and for solving problems. The first part of the unit was also important because it gave the students some contact with reasoning processes they were not familiar with as generalizing and expressing generality. Sharing different strategies was an important feature that contributed to enrich the general discussions. These moments, most especially, the discussion of the first task, were very important places to clarify of difficulties and negotiate meanings.

All the students, even those who most feared mathematics, got immediately involved in solving the open tasks. At the beginning their reasoning was supported by the geometric representation of the sequences but the other tasks allowed them to go further on their reasoning. Some of the students made some initial mistakes, that were discussed and clarified in the general discussions. However, most of the groups solved the tasks in an autonomous way, which gave them the trust and motivation they needed to solve the remaining tasks. This underscores the idea that these tasks yield multiple entry points to students with different ability levels in mathematics. The challenging nature of the tasks involved all students in developing their own strategies.

Working through exploratory and investigative tasks seems to have contributed towards: (i) developing a richer meaning for the algebraic language; (ii) widening the strategies to explore situations involving variables; (iii) using reasoning of an increasingly general nature; and (iv) expressing them using a more formal language. In this way, the work through the teaching unit seems to have contributed in an important way towards the development of students' symbol sense, yielding opportunities to strengthen their algebraic thinking. The exploratory and investigative tasks included in the unit, the work in groups and the classroom discussions matched the initial expectations, generating experimentation, autonomous work and most importantly, assuring that algebra was always a sense making activity. In this way, the students could construct new concepts and enlarge their algebraic knowledge and thinking.

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