A TRAJECTORY TO GENERALIZATION: THE TEACHER'S SUPPORT TO PUPILS' MATHEMATICAL INVESTIGATIONS IN THE CLASSROOM

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The aim of this paper is to analyze the teacher's role in supporting pupils' work in mathematical investigations, in particular, with one task that involves generalization, and the constraints the teacher faces when performing this role. From the results it appears that to understand the teacher's role in this situation it is necessary to take into account the nature of the previous tasks and how they related to this one, the teacher's perspectives about what counts as generalization and the different possible representations involved in the task, and, finally, the compatibility of the goals established by the teacher for the task and for supporting pupils.

1. INTRODUCTION

This research develops in the context of one teaching experiment that promoted mathematical investigations in the classroom with 8^{th} graders. Assuming that the teacher's role is particularly demanding in classes where pupils develop this kind of activity, it is important to understand how can teachers perform a different role and the challenges they face. In this paper I present the case study of one teacher working in the classroom with one task that involves generalization in the domain of algebra. Considering that the practice of algebra teaching still needs much more research (Kieran, 2007), two research questions are targeted: What is the teacher's role in supporting pupils' mathematical activity with numerical and algebraic investigations, namely when involving generalizing processes? What constraints does the teacher face when performing this role?

2. THEORETICAL BACKGROUND

2.1. Teacher's role regarding mathematical investigations

The concept of mathematical investigation as an activity in the learning of school mathematics constitutes an educational metaphor: it intends to bring to the classroom some of the characteristics of the genuine mathematical activity. The pupil should develop mathematics in the sense that he has to formulate questions and conjectures, test these, build justifications and refutations, and to argue with the teacher and colleagues. Teacher's role in promoting and nurturing this activity is a very relevant, complex and demanding one. It is necessary to support pupils on their work without leading them towards a desired outcome, which implies, to pose good questions to pupils, evoke relevant information, exhibit mathematical reasoning, and promoting the reflection by the pupils (Ponte, Brocardo & Oliveira, 2003).

The interactions that take place in the classroom, according to the Symbolic Interaccionism, can not be reduced to a sequence of actions and reactions, because each participant controls his/her action taking into account what he/she assumes as the others participants' understandings and expectations (Voigt, 1994). Even when the participants don't explicitly present their points of view, the mathematical meaning is continuously negotiated. The mathematical discourse involved in mathematical investigations in the classroom assumes a particular nature, very different from the ones where the pattern is: the teacher formulates one question whose answer she/he knows, the pupil answers, and finally the teacher validates it (Wood, 1994). Interaction patterns and processes of communication are mutually constituted by the teacher and the pupils but it has to be the teacher who starts defining new frameworks for the activity when pupils are not acquainted with that.

2.2. Pupils' generalizations

Generalizing is an intrinsic aspect of the mathematical activity, therefore its one of the highlighted processes in mathematical investigations in the classroom. As Steen states "mathematics is the science of patterns" (1988), and as so generalization about patterns and regularities can be an important entry point into the process of generalization. Research documents that pattern formulation can help pupils' introduction to algebra (Stacey & MacGregor, 2001). However, research also documents the pupils' difficulties when they start working with generalization in sequences, because the transition from the particular to the general takes time (Kieran, 2007). For instance, it is very common that pupils use an additive approach to sequences, by connecting consecutive terms of the sequence, or fail in distinguishing between the Growing rule and the Position rule (Warren, 2006). Working with different representations can enhance pupils' skills in generalizing, but research also shows that tabular representations of patterns may not help pupils to identify the "general relationships underlying patterns" (Kieran, 2007). Therefore the teacher needs to attend to these particularities of pupils' mathematical thinking and development, and to act accordingly.

3. METHOD

The research methodology adopted in this study is qualitative and interpretative and follows a case study design. The teacher's practices and her perspectives about her actions were analyzed. The researcher took part in the lessons as participant observer as she was introduced to pupils as one more teacher in the classroom and sometimes interacted with them to support their work. Data from the classroom were collected through video recording of the moments of whole class work and audio recording of all teacher's speech during the lesson. Additional data came from interviews and post-lessons reflections with the teacher. All these data were transcribed and analyzed by the researcher, and all the documents produced were read by the teacher.

The teacher, Isabel, has about 20 years of experience, teaches in this school for about a dozen years and mainly in the secondary level (10^{th} and 11^{st} grades). She has a strong mathematical background, having studied applied mathematics for five years. Her love for mathematics and for teaching makes of her a teacher who derives great satisfaction from the profession. That was one of the reasons to choose her as participant in this study.

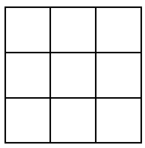
In this presentation I will focus on one of the four tasks that the teacher proposed during this class experiment with one 8^{th} grade class with 29 students. As it happens with the other lessons in the experiment, the teacher works with the class distributed in six groups and through three moments: (i) brief oral presentation of the task by teacher, (ii) independent small group work, and (iii) whole class presentation and discussion of pupils' achievements. Due to limitations of space I simply analyze here the teacher's role regarding the first two moments.

4. RESULTS: The case study of Isabel

4.1. Planning the lessons

Isabel regards this experiment as an opportunity for pupils and herself to develop new roles. They will be more independent, trying to discover things by themselves, "the 8^{th} grade's little investigators", as she calls them. She views her role as someone who is there to support pupils' activity but won't direct them to a certain strategy or answer. Isabel prepares herself for these lessons solving the tasks and sometimes suggesting some modifications. In the case of this task, "Squares with matches" (see Figure 1), she suggests to include the word "investigate" in the second question, because she fears that pupils will only look for a number, like in the first question.

Squares with matches



- How many matches are used in this square?
- Investigate how many matches are necessary to build any square of this type.

Figure 1 – Squares with matches

The strategy she uses to find the general expression is constructing a table, factorising the numbers on the sequence and gets: n x (2n+2). She realizes that pupils can use very different strategies which can be very rich from a mathematical point of view. The figure in the task is seen by her as a possible motivation to pupils to engage in the activity. As they were acquainted with writing a general expression for a sequence (a topic worked four months ago but with easier sequences), the teacher plans to use two lessons for this task, including the discussion of the results (in a total of 90 minutes).

4.2. Introduction to the task

In order to help the pupils to approach the task, Isabel recalls on that they have explored some months ago, involving a sequence of the areas of squares that was represented in a table, where the pupils got at a general rule for the sequence. The challenge says the teacher, is that in this task "you don't have any sequence: you will have to find it and to build it". It appears to be clear that what she intended for this "investigation" was that pupils come to a generalization using symbols.

4.3. Teacher's support to pupils' activity

Right from the beginning, Isabel realises that some groups don't understand, in the second question, what does it mean "any square of this type". She helps them to identify the sequence of squares starting with the one with one match on the side. In spite of the introduction, focused on finding the general expression for the sequence, the groups tended to follow a recursive strategy, perhaps because in the previous tasks in this experiment they looked for rule of construction relating the terms of the sequence (looking for patterns in the numbers, i.e for properties of the numbers) and as literature shows this is easier for them. Isabel cope with this suggesting pupils:

- *To display more steps*. In order to help them to recognize the law of construction of the sequence the teacher sometimes suggests to pupils to go for one or two steps further (n= 4 or 5). But usually that is not necessary since the pupils take the initiative in doing it.

- *To draw a table (a different representation).* This was suggested implicitly in the introduction done by the teacher. This hint didn't help most of the groups, since this to be effective the pupils need additionally to refocus their attention in two things: a "useful" factorization of the sequence terms and the relation between the term of the sequence and the respective order (instead of comparing consecutive terms). Sometimes this derives the attention of the pupils from the strategy they were trying because they don't see its relation with the teacher's hint. What happens then is that most groups still look only to the numbers in the sequence and try to find a relation between consecutive terms. The teacher doesn't want to suggest the kind of factorization she used because she thinks that her role is only to support and not direct the pupils' explorations towards one process.

- *To establish a relation between the term and its position.* When pupils are concentrated in finding a property in these numbers (for instance, some pupils conjecture that each number is the double of the previous one, because they notice that 24 appears in the sequence after 12), Isabel many times says to them: "Try to relate these [the numbers in the sequence] with the number of matches on the side of the square". After a relative long period the teacher explicitly suggests to some groups to do some computation between the term and its position. This leads some groups to divide the numbers in the sequence by the respective order, obtaining the sequence of even numbers, but in some cases the pupils didn't know what to do with these, because they could not find a relation between the terms and their respective order (like the double or the triple, as they expected).

- *To look for familiar sequences in the numbers*. After finishing the first lesson, only one group succeeded in writing a general expression for the sequence, exactly using the same strategy as the teacher. Therefore she decides to start the second lesson by remembering some of the sequences they worked some months ago (natural numbers, even numbers and multiples of four) and recalling the meaning of the order and the term in a sequence. One pupil immediately establishes a relation between this hint and the work they have been doing, recognizing the sequence of even numbers starting in four. Interestingly one of the groups even after having factorized the terms as the product of the order by the sequence of the even numbers starting in four does not identify it as that sequence and still focus the attention on consecutive terms. The teacher's questioning here becomes crucial:

Teacher- Haven't you find an expression?

Daniel – I found something but still we had to come back and then I thought that is useless because…

Teacher – You always had to get to the previous term, is it?

Daniel – To get the number of matches, we multiply by four and then we multiply by plus two, by six, then by eight, by ten and so on… but it doesn't do…

The teacher now asks them to write down the products Daniel mentioned, and one of the girls in the group refers that those are even numbers. Relying on this comment the teacher helps them to recognize the two known sequences on the product.

- *To use a proper variable*. In some groups the teacher mentions explicitly the need to choose a variable for the general expression of the sequence. She recalls them that, as the order is a natural number, *n* can be an appropriate letter to use in the expression.

4.4. Teacher's underachievement in supporting pupils' activity

The goal established by the teacher for this task it seems to be the generalization of the sequence using symbols, as I mentioned before. However, her role in supporting pupils' activity in generalizing didn't succeed as expected in some instances, as she:

- *Induces the pupils towards a recursive approach.* The teacher usually suggests to the groups to establish a relation between the term and its position, however in the dialogue with them she sometimes, focus their attention in a possible relation between consecutive terms. In this group they have the terms of the sequence until n=5.

Teacher- Now you have to look at the "behaviour" of the sequence, one that from four goes into 12, from 12 goes to 24, from 24 to 40, from 40 to 60…

Ana – It is always the double!

Fábio – No.

Teacher – The double of what?

Beatriz – No it is not.

Teacher – Look at this "behaviour". There is always something there that is common. There is always something that we do when we pass from…

Fábio – (interrupting) Four times three, 12.

Beatriz – They're always multiples of four.

What the teacher intended with this support was that they notice what was happening to the terms in a functional way, but the pupils didn't interpret that way. They just look for patterns in the numbers, relating consecutive terms.

- *Uses indistinctively law of construction and general rule.* At the same time, the teacher seems not to take into account the importance of the use of rigorous terms when she refers to the intended outcome of the pupils' activity within this task. Sometimes she asks to the one group, simultaneously, for the law of construction of the sequence and for the general expression, apparently referring herself to the same object.

- *Does not stimulate the use of symbols in a recursive reasoning or rule.* Some groups identify the sequence in a general but recursive way, describing the rule. For instance one group states "We sum the difference between the two previous ones and add four to obtain the following", and after a period of work presents "$n + (n-n_1)+4$", meaning $u_{n+1}=u_n+(u_n-u_{n-1})+4$, with n natural. The teacher verifies the meaning of the recursive rule they presented but doesn't discuss with them their use of the letters. A similar situation happens with one group that notices that the difference between consecutive terms is the sequence 4n+4. However she doesn't help them to write down the recursive rule, even without symbols.

- *Does not focus on the figure.* The majority of the groups only look at the figure in the initial phase to count the number of matches in an organized manner. After that they centred their exploration on the numbers they obtained. The teacher never tried to focus the pupils on the way they counted the number of matches to come to the general expression. One pupil arrived at the expression "$\ell x(\ell-2+\ell)+(\ell x4)$" ($\ell$, the number of matches on the side of the square) counting the matches in the perimeter of the square and the matches inside of it, but the teacher had no intervention on this.

- *Does not support different factorizations.* The teacher is not able to help to generalize those pupils who factorized the numbers in the sequence in a different way from the one she had. For instance, one group gets the sequence: 4x1, 4x3, 4x6, 4x10… In this case she leaves the group without giving a suggestion for obtaining the general expression.

5. DISCUSSION AND CONCLUSIONS

Pupils' first approach to the task is to compare consecutive terms of the sequence, some of them explaining the relation in a general rule. They reveal great difficulty to depart from this kind of reasoning to the one that expresses the functional relation. When the teacher recalls some of the sequences of special numbers they worked some time ago, some groups succeed to write the general expression (sometimes with some direct help from the teacher).

5.1. The teacher's support

In the previous section I identify five aspects of the teacher's role in supporting the pupils' activity. These can be seen as having mainly the influence of: (i) the strategy the teacher employed to solve the task, (ii) her expectations about the pupils' knowledge on sequences of numbers, and (iii) the intended outcome, a specific understanding about generalizing as presenting a general symbolic expression. May the teacher have other kind of experience in teaching algebra in the middle grades, this role could have been very different. It is clear here that the teacher's support in this task has been mainly focused on "one route to generalization". Probably her main experience in teaching in secondary level didn't question her substantially about how to support pupils to go from the particular to the general.

5.2. Constraints to the teacher's role

The intended support for pupils' activity sometimes didn't succeed as expected, what might be linked to several aspects, as follows.

The nature of the previous tasks, based upon patterns and regularities (mainly in terms of numbers' properties), created a false expectation in pupils that when they operate on the numbers of the sequence somehow they will come to that kind of regularity. The teacher, apparently, did not consider this influence on pupils, and in some moments still reinforced the perspective of looking to the sequence as a list of numbers with a particular property.

The challenge of this task for pupils is much bigger than that of the previous ones (when they worked with sequences of numbers or in the experiment). The teacher didn't take this into account when she prepared herself for this lesson. Even when she tries to push the pupils to the strategy she used, she misses the fact that it is more complicated to reason upon the product of two sequences than of only one.

The intended goal for the task and the role the teacher wants to perform are mismatched. The teacher's role is informed by the meaning that she attributes to the task's nature. Being integrated in an experiment with mathematical investigations, the teacher's perspective is that she should not be directive in her support to the groups. However, this task, that could be an investigation, was transformed by teacher's intentions into a problem of finding the general expression for one sequence. In here the teacher faced the dilemma of putting the emphasis on the kind of activity that she intended pupils to develop (an investigative one) or on the activity's outcome. This situation is particularly visible with those pupils that factorise the sequence in a different way but don't receive any suggestion from the teacher to come to the generalization.

The notion of what constitutes a desirable generalization is too limited. For this teacher the work with sequences appears as an introduction to the operations with polynomials. Therefore the functional dimension is less focused by the teacher who, as mentioned above, centres on the identification of known sequences of numbers to formulate the general symbolic expression. When the teacher suggests the use of variables (for instance, n for the natural numbers) these appear more as letters that represent particular values than relationships (Becker & Rivera, 2005). On the other hand taking into consideration that we are talking about middle school pupils, it could be more advantageous to give them the opportunity to formulate in words or symbols the rules they found, and support them to be rigorous in that. The teacher's experience with the secondary school mathematics, sustained by a sophisticated mathematical knowledge, may have an influence on her perspective about what is generalization that is worthwhile for these pupils.

The role of different representations in supporting pupils' trajectory to generalization is not completely perceived by the teacher. Pupils could have formulated the generalization from the visual representation of the sequence, but the teacher didn't call their attention to that. Research is showing that pupils who use the visual representations employ strategies with a focus in the relationships (Becker & Rivera, 2005), but usually teachers are not acquainted with this.

5.3. Concluding remarks

This research centred on the mathematics classroom practice allows us to understand the complexity of the teacher's role when pupils work on tasks that involve generalizing and induction processes. The teacher's knowledge to handle these can't be developed simply by learning about it, nor from working with these in the classroom. Reflecting on this kind of experiences with (significant) others may help them to develop the necessary expertise to perform their roles within these demanding classroom scenarios. Teacher education faces the challenge of providing these significant others in relevant contexts of practice.

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