SECONDARY SCHOOL STUDENTS' UNDERSTANDING OF THE CONCEPT OF FUNCTION[1]

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Abstract: This paper aims to present secondary school students' understandings of the concept of function, in an investigative and problem solving approach, and to identify some students' difficulties concerning that concept. It begins with a presentation of the ways in which researchers describe the symbol sense, the understanding of the concept of function, and the approaches with problem solving and investigative tasks. Subsequently, it presents the qualitative methodology used in the study, and finally its results. The students were able to manipulate the symbols, and operate with them, but this was not sufficient for the structural understanding of a function. Also, this study underscores that an investigative and problem solving approach is a rich way to learn the concept of function.

Introduction

The concept of function is one of the basic mathematical concepts. The students face many difficulties when they attempt to understand this concept and when they need to use the chain of symbols that are connected with it. The concept of function is a crucial subject of the Portuguese curriculum of the secondary school (10th, 11th, and 12th grades). This paper presents some of the preliminary results of a study developed with students of a grade 11 class, in the topic Functions II. It presents some secondary school students' understandings of the concept of function, in an investigative and problem solving approach, and identifies some students' difficulties concerning that concept.

Symbol sense

Algebraic thought, for Ponte (2005), includes the capacity to work with algebraic calculation and functions, and the capacity to deal with many other mathematical structures, as well as with the capacity of using the symbols in the interpretation and resolution of mathematical problems or other domains. In algebraic thought is still included symbol manipulation and symbol sense. This, for Arcavi (2005), consists in the capacity to interpret and to use, in a creative way, mathematical symbols in the description of situations and in problem solving, as the choice of symbols, the flexible manipulation skills, and symbols in context. On the other hand, Rojano (2002) reminds us of the importance of establishing how the meaning variation of mathematical symbols during the transition from arithmetic to algebra represents an obstacle in the subject's evolution toward the acquisition of algebraic language. These differences in meaning of the same symbols and symbol chains present serious difficulties for secondary school students in the learning of algebra, challenging the old idea that algebra could be conceived, for teaching purposes, as "an extension of arithmetic" (p. 145). Rojano (2002) says, additionally, that the majority of students in secondary school are not able to connect by themselves the knowledge domains that constitute manipulative algebra on the one hand and instrumental algebra for the problem solving on the other. The transition from particular to the general also has been discussed, namely the haste to symbolization, during the accomplishment of the tasks of generalization in the classroom. Inadequately, teachers

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usually have an apparent tendency in teaching to abbreviate the process, and by this way they do not give to the students the opportunity to formulate the algebraic equation for the stated problem.

For Chazan, D. and Yerushalmy (2003), school algebra might be thought of as cut and dried. To these authors, symbol chains involved in the problems of algebra evade a simple classification, which implies that the students cannot know the concrete methods they may choose to solve the specific problems, or that they may choose incorrectly a method to solve the problems. This means that it is necessary to develop a curriculum that does more than teaching specific methods to solve certain types of problems, where "the instruction should connect with students' experience and build on the resources and strengths present in the conceptions they bring to school" (p. 133).

In this study we adopt the concept of symbol sense given by Arcavi (2005). We also think that it is necessary that students obtain a feeling about the most appropriate methods to work with the symbol chains and that they should appreciate the meanings of those methods.

The understanding of the concept of function

The concept of function is one of the fundamental mathematical concepts. It is extraordinary in the diversities of its interpretations and representations. However, the students face many difficulties when they try to understand it. Chazan, D. *et al*. (2003) say that usually functions are conceptualized as a special type of relation. In fact, every linear equation can be written as an equivalent equation which is also a linear function in one variable. To create a graphic of one linear equation with two variables it may be useful to write it as a linear function with one variable – those authors emphasise the connections between the graphics and the expressions as eventual benefits to the understanding of the existing equivalences and the differences. In this sense, they pose such questions: How to develop the curriculum to help the students to promote a sense of the different types of symbols and of the various uses of the notions such as variable, equal sign, and the system of cartesians coordinates? Does it matter the way the students learn those notions? Does the introduction sequence matter?

Sajka (2003) says that one of the students' difficulties in understanding the concept of function stems from its dual nature. In fact, and according with Sfard (1991), the function can be understood in two essentially different ways: i) structurally – as an object; and ii) operationally – as a process. In the first, the function is a set of ordered pairs, and from the operational way it is a computational process or well defined method for getting from one system to another. Those two ways of understanding functions, although apparently ruling out one another, however, should complete each other and constitute a coherent unity – like two sides of the same coin (p. 230). For example, $f(x)=2x+3$ tells us two things at the same time: how to calculate the value of the function for particular arguments (evoking the process) and it encapsulates the whole concept of function for any given argument (thus presenting the object). So, we can say that $f(x)$ represents both the name of a function and the value of the function f. And, in the context of functions, sometimes when we write y, we are referring to a certain value of the function: at other times we are referring to the ordinate of a certain point in the coordinates system, and yet in other times we are referring to an argument. The interpretation depends on the context, which can confuse a non-advanced student. This notation of function is ambiguous and promotes some difficulties among students. For Sajka (2003), the causes of students' symbols difficulties also depend on the contexts in which the symbols are worked in mathematics classes, and on the teachers' limited choices of mathematical tasks. For this author, the concept of function is often linked to the concept of formula, and sometimes the students connect the concept of function to the graphic process, where a formula is necessary to draw it.

So it is very important for teachers' professional practice to take into account the existence of this ambiguous notation, like the function one.

The mathematics class with exploration and investigative tasks

For several authors to learn mathematics is to understand its nature. Goldenberg (1999) says that it is very important to develop the students' activity because one of the aims of the mathematics education must be to make the students learn how mathematicians discover mathematical methods and facts. For that, it is fundamental that students spend time with problem solving, exploratory, and investigative mathematical tasks. The aim is for the students to learn how to be astute researchers, and for that, it is necessary that they do investigation. This idea is clear in many curricular orientations in several countries. For NCTM (2000), the key to the promotion of the students' performance in a certain domain, like school algebra, is not the creation of an ever more elaborate and finely-tuned set of procedures, but rather by changing the nature of instruction. One cannot ignore students' conceptions and it is necessary to confront students' misconceptions. Although the focus of learning would not be exclusively problem solving, exploratory and investigative tasks (there are others, such as exercises), these one can promote the students' engagement in creating and discovering genuine mathematical processes (Pereira, 2004; Ponte *et al*, 1998; Teixeira, 2005).

The teacher's role in promoting the students' mathematical activity is crucial. The students' interest will be stimulated by the mathematical tasks selected by the teacher, and by the situations and contexts that the teacher promotes in the class, as well as by their capacity to develop and to lead the students' activity with success. It will be the mathematical tasks and situations that give the opportunity to the students to develop their own algebraic thinking. Also, to get a good integration of problem solving, exploratory and investigative tasks the teacher needs not only to mobilize theories and techniques but still to mobilize his conceptions, feelings and his practical knowledge (Saraiva, 2001).

Methodology

A three month-long study (February – May, 2007) was conducted in a secondary school: The *Functions II* course (K11 level), which made use of graphing calculators, in mathematics class with 24 students. In accordance with the pedagogical proposal previously elaborated, taking into account the curricular requirements and the problem of the study, the subject of *Functions II* was developed with resource to the resolution of problems, exploratory and investigative mathematical tasks (some of them with the use of the graphical calculators), along with moments of verbal exposition of programmatic contents and of the resolution of exercises of more usual nature. Many times the students worked in groups. The teacher of the group is the second author of this paper. The secondary school is situated in Covilhã, Portugal.

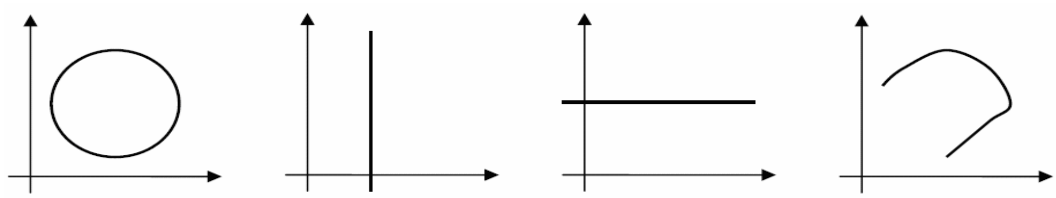
In this study a qualitative research methodology was followed. The data were collected through i) the oral comments of the pupils – working in groups and in the moments of the collective discussion, registered by the teacher/researcher in her diary; ii) two questionnaires that the students answered – the first one at the beginning of the course and the other one at the end of the instruction of the subject *Functions II*; iii) the written tests taken by the students; and iv) the students' exploration of some tasks (*Questionnaire* – to identify the previous knowledge students have on functions; *Transformations of rational functions* – to explore the influence of the parameters *b*, *d* and *a*, respectively, using the graphic calculator, in the functions family of the type $y = \frac{b}{x}$ $(b \neq 0)$, $y = \frac{1}{x+d}$ and $y = a + \frac{1}{x}$; *Operations with functions* – to introduce the addition, difference and product operations with functions). The analysis started since the beginning of the study, having, however, a more intense phase after the collection of all the data. These, in a first phase, were grouped by instrument of collection of data (diary; questionnaire; ….), and later by categories (*functions* - definition; representations; connections; symbols; difficulties -; *other aspects of algebra* - algebraic manipulation; symbols; difficulties -; *resolutions and original strategies of the students*). After this categorization a transversal analysis was made.

Results of the study

In the first lesson of the subject *Functions II* a questionnaire was supplied to the students. The main purpose was to identify the previous knowledge students have of functions. From the analysis of their answers, some aspects deserved special attention. In a first phase (figure 1) students were to identify, among the given graphs, the one(s) that represented a function and, afterwards, to say by their own words, what they understood by function.

1. Choose the graphic(s) that represent a function. Justify.

(A) **(B)** **(C)** **(D)**



2. Say, using own words, what is a function.

Figure 1 - Extract of the questionnaire

Only seven of the twenty-four students identified, rightly, the option (C); the great majority (sixteen students) chose the options (B) and (C); only one student opted for the graph (A).
From the data we can say that some students do not connect successfully the definition of function they write with their choice of the graphics that represent a function. For example, the definition "In a function, to each object corresponds one and only one image" is associated to the choice of both options – (B) and (C). Also, some students present the statement "there are no repeated points" and choice the graphic (C). It seems that, for them, the definitions are only words.

The relation of dependence between the variables was one of the aspects highlighted by definitions given by the students, as we can see in the next answered: "In a function there is a dependent variable (y) and an independent variable (x)"; in another answer, this idea was enhanced, as we can see: "y changes in function of x and each x originates one and only one value for y". However this relation of dependence seems to be chained to the symbols y and x. In fact, in the answers to the question 4 of the questionnaire (figure 2) – "where a graph was given showing the variation of the amount of water in a dam, throughout one year, in function of the time, measured in days, and students to identify the dependent and the independent variables" – some students attribute the symbol *x* to the number of days (*t*), and the symbol *y* to the existing water (*Q*).

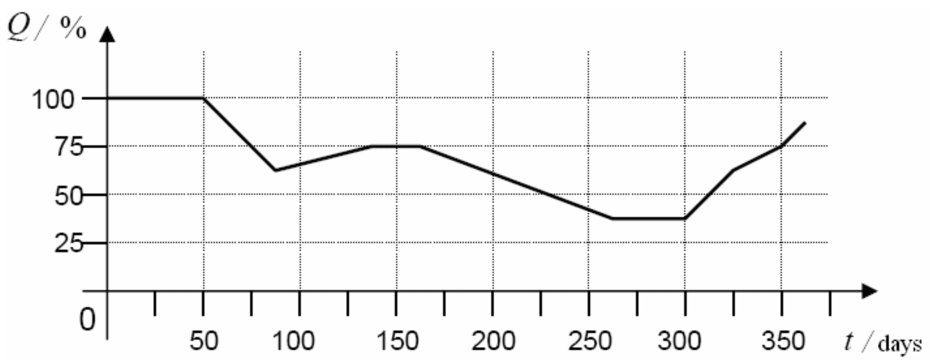
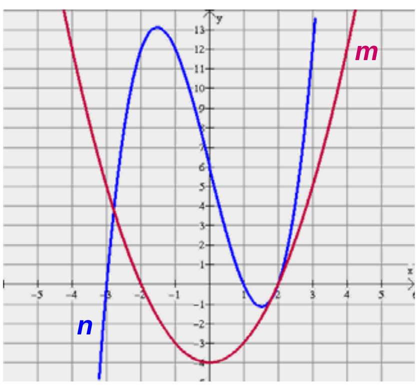


Figure 2: Extract of the first questionnaire

For the students, the variable concept has a *static* meaning. They did not convey an idea of variation, as we can see in the answers to this question, where the students write "days" (and not "number of days") as the independent variable, and "water" (and not "amount of water") as the dependent variable.

The friendship that students made with the variables influenced their performance in other tasks throughout the study. In the first part of the task "Operations with functions", two functions were given, $\boldsymbol{f}$ and $\boldsymbol{g}$, expressed analytically, and the students must identify the analytical expressions of the functions $(\boldsymbol{f}+\boldsymbol{g})$, $(\boldsymbol{f}-\boldsymbol{g})$ and $(\boldsymbol{f}\times\boldsymbol{g})$, its graphics, and its ranges. Afterwards, the students were invited to choose other pairs of functions $\boldsymbol{f}$ and $\boldsymbol{g}$, and to operate with them to find the relationship between the $\boldsymbol{f}$ and $\boldsymbol{g}$ ranges. As it was recorded in the students' exploration, in an intuitive way, they associated the addition, the difference, and the product of functions to the addition, difference and product, respectively, of the functions that were supplied to them or those that they chose and carried through with relative effectiveness the necessary calculations. Nevertheless, the great majority of the students had difficulty to define a strategy to solve the second part of the task (figure 3). In fact, when students faced the inexistence of the analytical expression of these functions they did not collect the information supplied in graphical terms, and they didn't know what to do to determine the values of $(\boldsymbol{m}\times\boldsymbol{n})(-1)$ and of $(\boldsymbol{m}\times\boldsymbol{n})(3)$. The students' dialogue with the teacher, organized in groups, was fundamental for them to understand that the required information was, now, presented graphically.

The figure shows the ***m*** and ***n*** functions, of 2nd degree and of 3rd degree, respectively.



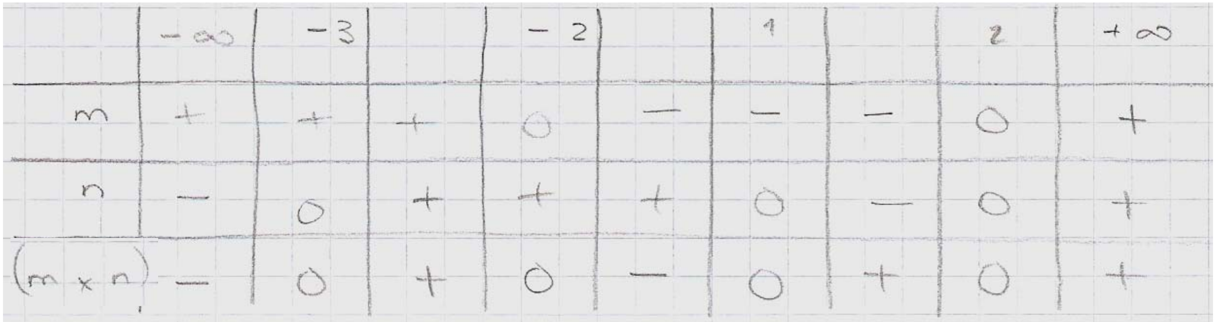
By observation, say:

- which is the value of $(\boldsymbol{m}\times\boldsymbol{n})(-1)$ and of $(\boldsymbol{m}\times\boldsymbol{n})(3)$;
- the variation of signal of the function $(\boldsymbol{m}\times\boldsymbol{n})$

Figure 3 – Extract of the task "Operations with functions"

In this case, the students' confidence of an apprehended analytical routine was an obstacle for them to search the two concreteness of the variable *x*. One student wrote: "[in the first part of the task] it was easier, because we had *x* and the functions were given, and we knew already what to do…".

In spite of the initial difficulties, the students' exploration of the variation of signal of the function $(\boldsymbol{m} \times \boldsymbol{n})$ shows us that they were able to understand, in a general way, what to search in the functions ***m*** and ***n*** and to recall their previous knowledge in order to organize the answers (figure 4).



	$-\infty$	-3		-2		1		2	$+\infty$
m	+	+	+	0	−	−	−	0	+
n	−	0	+	+	+	0	−	0	+
$(m \times n)$	−	0	+	0	−	0	+	0	+

Figure 4 – The signal of the function $(\boldsymbol{m} \times \boldsymbol{n})$ - task "Operations with functions"

Throughout this study the students were involved in their own explorations, following their intuitions. In the task "Transformations of rational functions" it was asked of the students to explore the influence of the parameters *b*, *d* and *a*, respectively, using the graphic calculator, in the functions family of the type $y = \frac{b}{x}$ $(b \neq 0)$, $y = \frac{1}{x+d}$ and $y = a + \frac{1}{x}$. Two students devoted some time to explore curiosities with the functions of the type $y = \frac{b}{x}$. In their report, written in the class, they wrote: "We thought that the "points of curvature" appeared in the same coordinates; we thought that the "points of curvature" belonged to the straight line $y = x$".

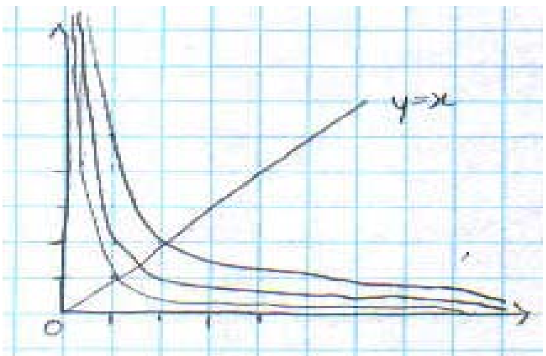


Figure 5: The "points of curvature"

Throughout the exploration of this conjecture, the students started to call the "points of curvature" as "points of change". Latter, this denomination was used in the whole class, in the collective discussion. Their reasoning was supported by the sketch of the graphics (fig. 5). The symmetry of the hyperbole's branches were analysed by the students using their intuition and they expressed that with proper words, identifying it analytically (*y*=*x*). However, they refuted their formulated conjecture because they used a wrong reasoning based on the fact that “point of change” of the function $y = \frac{1}{x}$ appears at (1,1), and they thought that in the function $y = \frac{2}{x}$ the “point of change” would be at (2,2), and in the function $y = \frac{3}{x}$ it will be at (3,3), and so on.

In the discussion class, where the conclusions were validated, this conjecture was analysed together and the situation was explored using the table of values of the graphic calculator. Because it was not easy to identify the coordinates of the “point of change”, the situation was analysed analytically. The teacher/researcher suggested to find those coordinates through a system of equations ($y = x$ and $y = \frac{b}{x}$, $b > 0$) and it was possible to conclude that the “points of change” are the points whose coordinates are the form $V_1\left(\sqrt{b}\,;\sqrt{b}\right)$ and $V_2\left(-\sqrt{b}\,;-\sqrt{b}\right)$.

The collective negotiation of meanings allowed the students to discuss their ideas, and to clarify and to share opinions as to the personal understanding of the mathematical concepts. In fact, one of these two students explained the way they follow to call to those points as “points of change”. To the student, the “points of change” change the relation between the objects and the images. To the left of them, the objects are smaller than the images and after the objects are larger [b>0]. This collective discussion also presented good opportunities for the students to express their difficulties and it was important to identify some misconceptions in the students’ learning. In fact, searching analytically the “points of change” [now with b<0], the students’ interventions evidence one limited vision of the concept of variable and its number value. Easily the students identified the symmetry axis of the hyperboles of the form $y = \frac{b}{x}$, $b < 0$, as the bisector of the even quadrants and with the straight line that contains this bisector. Yet, they said that the analytical expression was $y = -x$. However, in the resolution of the correspondent system it was not easy to convince the students that the equation $x^2 = -b$ is possible and the solution is $x = \sqrt{-b} \vee x = -\sqrt{-b}$. Their reaction was as follows: “But it is not possible! There are no roots of negative numbers!” It was necessary to use many examples so that they could see that the symmetric of a negative number is a positive one.

Conclusion

This preliminary study shows us that the students evidence a good skill and a good capacity to operate algebraically with the analytical expressions of a function.

However, the definition of function was memorized by some students, but most of them were not able to connect the words they write, such as “…to an one object corresponds one and only one image”…, with the graphic representation of one function, by choosing graphic representations that do not represent one function.

Students’ capacity to operate algebraically with the analytic expressions, for example, adding and multiplying two functions, was not sufficient for the search of two concretizations of the variable with the graphic representation. The analytic expressions of each one of the two functions were necessary to solve the initial question. Clearly these students associated the concept of function to the analytic representation and, as Sajka (2003) says, the students’ capacity to manipulate the symbols, and to operate with them, is not sufficient for their structural understanding of a function.

This study makes evident the profit that students had made as for the symbol sense, as a direct consequence of the explorations that they had made around the graphical representations of functions and the study of its properties. The collective discussion of a misconception ("square root of *-b* has no meaning") allowed the students to extend the interpretation of letter as unknown.

This study reinforces the importance of the research on the mathematical theme of functions, according to several authors (Arcavi, 2005; Ponte, 2005; Sajka, 2003; Sfard, 1991; Rojano, 2002). Despite mathematical notation ambiguity, the students' mathematical activity, using problem solving and investigative mathematical tasks, in a classroom context with good interaction between the students and between them and the teacher, allowed to improve meaningful learning on functions and, also, to identify the difficulties that the students faced.

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