Deconstructing the Mathematics Curriculum: Telling Choice from Nature

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Mathematics education is an institution claiming to provide the learner with well-proven knowledge about well-defined concepts applicable to the outside world. However, seen from a skeptical sophist perspective wanting to tell nature from choice, three questions are raised: Are concepts grounded in nature or forcing choices upon nature? How can an ungrounded mathematics curriculum be deconstructed into a grounded curriculum? Does mathematics education mean pastoral patronization of humans, or anti-pastoral enlightenment of nature?

The Background

Enlightenment mathematics treated mathematics as a natural science. Exploring the natural fact many, it induced its definitions as abstractions from examples, and validated its statements by testing deductions on examples (Kline, 1972, p. 398). Inspired by the invention of the set-concept, modern mathematics turned Enlightenment mathematics into a purely deductive 'metamatics' that by defining its concepts as examples of abstractions, and by proving its statements as deductions from meta-physical axioms, needed no outside world and becomes entirely self-referring. However, a self-referring mathematics soon turned out to be an impossible dream. With his paradox about the set of sets not being a member of itself, Russell proved that using sets implies self-reference and self-contradiction known from the classical liar-paradox 'this statement is false' being false when true and true when false: definition $M = \{A \mid A \notin A)\}$, statement $M \in M \Leftrightarrow M \notin M$.

Likewise, without using self-reference it is impossible to prove that a proof is a proof; a proof must be defined. And Gödel soon showed that theories couldn't be proven consistent since they will always contain statements that can neither be proved nor disproved.

Being still without an alternative, the failing modern mathematics creates big problems to mathematics education as e.g. the worldwide enrolment and justification problems in mathematical based educations and teacher education (Jensen et al, 1998); and 'the relevance paradox formed by the simultaneous objective relevance and subjective irrelevance of mathematics' (Niss in Biehler et al, 1994, p. 371). To design an alternative, mathematics should return to its roots, multiplicity, guided by a kind of research able at uncovering hidden alternatives to choices presented as nature.

Anti-Pastoral Sophist Research

Ancient Greece saw a struggle between two different forms of knowledge represented by the sophists and the philosophers. The sophists warned that to protect democracy, people should be enlightened to tell choice from nature in order to prevent patronization presenting its choices as nature. To the philosophers, seeing everything physical as examples of meta-physical forms only visible to the philosophers educated at Plato's academy, patronization was a natural order when left to the philosophers (Russell, 1945).

The Greek democracy vanished with the Greek silver bringing wealth by financing trade with Far-East luxury goods as silk and spices. Later this trade was reopened by German silver financing the Italian Renaissance; and by silver found in America. Robbing the slow Spanish silver ships returning on the Atlantic was no problem to the English; finding a route to India on open sea was. Until Newton found out that when the moon falls to the earth as does the apple, it is not obeying the unpredictable will of a meta-physical patronizor only attainable through faith, praying and church attendance; instead it is following its own predictable physical will attainable through knowledge, calculations and school attendance.

This insight created the Enlightenment period: when an apple obeys its own will, people should do the same and replace patronization with democracy. Two democracies were installed, one in the US, and one in France. The US still has its first republic; France now has its fifth. The German autocracy tried to stop the French democracy by sending in an army. However, the German army of mercenaries was no match to the French army of conscripts only too aware of the feudal consequences of loosing. So the French stopped the Germans, and later occupied Germany.

Unable to use the army, the German autocracy instead used the school to stop enlightenment spreading from France. Humboldt was asked to create an elite school, and using Bildung as counter-enlightenment he created the self-referring Humboldt University (Denzin et al, 2000, p. 85).

Inside the EU the sophist warning is kept alive only in the French postmodern or post-structural thinking of Derrida, Lyotard and Foucault warning against patronizing categories, discourses and institutions presenting their choices as nature (Tarp, 2004). Derrida recommends that patronizing categories be ‘deconstructed’. Lyotard recommends the use of postmodern ‘paralogy’ research to invent alternatives to patronizing discourses. And Foucault uses the term ‘pastoral power’ to warn against institutions legitimizing their patronization with reference to categories and discourses presenting their choices as nature.

Anti-pastoral sophist research doesn’t refer to but deconstruct existing research by asking ‘In this case, what is nature and what is pastoral choice presented as nature, thus covering alternatives to be uncovered by anti-pastoral sophist research?’ To make categories, discourses and institutions anti-pastoral they are grounded in nature using Grounded Theory (Glaser et al, 1967), the natural research method developed in the American enlightenment democracy. As to the grounding of mathematics in the Enlightenment century, Morris Kline writes (Kline, 1972, p. 399)

> The enthusiasm of the mathematicians was almost unbounded. They had glimpses of a promised land and were eager to push forward. They were, moreover, able to work in an atmosphere far more suitable for creation than at any time since 300 B.C. Classical Greek geometry had not only imposed restrictions on the domain of mathematics but had impressed a level of rigor for acceptable mathematics that hampered creativity. The seventeen-century men had broken both of these bonds. Progress in mathematics almost demands a complete disregard of logical scruples; and, fortunately, the mathematicians now dared to place their confidence in intuitions and physical insights.

A Search Agenda

Confronted with the traditional mathematics curriculum wanting to provide the learners with well-proven knowledge about well-defined concepts applicable to the outside world (NCTM, 2000), anti-pastoral sophist research wanting to uncover hidden alternatives to pastoral choices presented as nature could raise three questions:

- Are concepts grounded in nature or forcing choices upon nature?
- How can an ungrounded mathematics curriculum be deconstructed into a grounded curriculum?
- Does mathematics education mean pastoral patronization or anti-pastoral enlightenment?

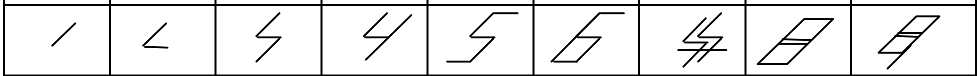
Thus the first question to ask is: are the roots of mathematics in nature or in human choices?

The Natural Roots of Primary School Mathematics

In primary school, the natural root of mathematics is double-counting the physical fact many.

Iconizing Many

Numbers coming from counting differentiates between degrees of many. 1.order counting means counting in 1s rearranging sticks to form an icon. Thus the five-icon 5 contains five sticks if written in a less sloppy way. In this way icons are created for the numbers until ten that becomes a very special and strange number having its own name, but not its own icon.

I	II	III	IIII	IIIII	IIIIII	IIIIIII	IIIIIIII	IIIIIIIII
								
1	2	3	4	5	6	7	8	9

2.order counting is counting by bundling&stacking using icon-bundles. First we bundle the sticks in e.g. 3-bundles, in 3s. Then we stack the total in two stacks: a stack of 3s, and a stack of unbundled singles. The stacks may then be placed in a left bundle-cup and in a right single-cup. In the bundle-cup a bundle is traded, first to a thick stick representing a bundle glued together, then to a normal stick representing the bundle by being placed in the left bundle-cup. Now the cup-contents is described by icons, first using cup-writing 2)1), then using decimal-writing to separate the left bundle-cup from the right single-cup, and including the unit 3s, T = 2.1 3s.

Thus a total of 7 1s is bundled&stacked as 2 3s + 1 1s, or as 2)1), or as 2.1 3s.

I I I I I I I -> III III I ->	III III I	-> ~~III~~ ~~III~~) I) -> ▮▮) I) -> II) I)
Or with icons:		-> 2 3s + 1 1s -> 2x3 + 1x1 -> 2)1) -> 2.1 3s

Later also bundles are bundled, calling for a new cup to the left. Thus 4 5s can be rebundled in 6 3s and 2 1s, i.e. as 6)2), where the 6 3-bundles can be rebundled into two 3-bundles of 3-bundles, i.e. as 2))2 or 2)0)2), leading to the decimal number 20.2 3s: III III) II) -> II)) II).

And 4 8s can be rebundled in 1 3-bundle of 3-bundles of 3-bundles, and 1 3-bundle, and 2 1s, i.e. as 1))1)2) or 1)0)1)2), i.e. as the decimal number 101.2 3s:

IIIIIIII IIIIIIII IIIIIIII IIIIIIII -> III III III III III III III III III III II -> ~~III~~ ~~III~~ ~~III~~ ~~III~~ ~~III~~ ~~III~~ ~~III~~ ~~III~~ ~~III~~ ~~III~~ II

Iconizing Counting

Operations iconize the processes involved in counting by bundling&stacking. Taking away 4 is iconized as –4 showing the trace left when dragging away the 4. Taking away 4s is iconized as /4 showing the broom sweeping away the 4s. Building up a stack of 3 4s is iconized as 3x4 or 3*4 showing a 3 times lifting of the 4s. Juxtaposing a stack of 2 singles next to a stack of bundles is iconized as + 2 showing the juxtaposition of the two stacks. And bundling bundles is iconized as ^ 2 showing the lifting away of e.g. 3 3-bundles reappearing as 1 3x3-bundle, i.e. a 1 3^2-bundle.

Numbers and Operations Form Formulas for Prediction

Now numbers and operations can be combined to calculations predicting the counting results by two formulas. The 'recount-formula' T = (T/b)*b tells that the total T is counted in bs by taking away bs T/b times. Thus recounting a total of T = 3 6s in 7s, the prediction says T = (3*6/7) 7s. Using a calculator we get the result 2 7s and some leftovers. These can be found by the 'rest-formula' R = T – n*b telling that the rest is what is left when the full stack is taken away: R = 3*6 – 2*7. Using a calculator we get the result 4. So the combined prediction says T = 3*6 = 2*7 + 4*1. This prediction holds when tested: IIIIII IIIIII IIIIII -> IIIIIII IIIIIII IIII.

Since formulas can be used to predict the result of counting, the scientific method using mathematical formulas for predictions to be tested can be introduced already in primary school.

Predicting and Practising Recounting

The recount-formula and the rest-formula now enables predicting and practising double-counting over and over, turning out to being the leitmotif of mathematics. First, only iconized numbers are used as bundle-size, recounting 3 5s in e.g. 7s, but not in tens that is postponed as long as possible. In this ten-free zone it becomes possible to introduce the core of mathematics using 1digit numbers only (Zybartas et al, 2005). The CATS-approach, Count&Add in Time&Space, is one example of a grounded approach to mathematics as a natural science investigating the natural fact many when counting by bundling&stacking, and when using double-counting at all school levels (Tarp, 2008).

Including ten as bundle-size means going on from 2.order counting, using bundles with both a name and an icon, to 3.order counting, using the bundle-size ten having a name, but not an icon.

Preparing for 10

Before introducing ten as 10, i.e. as the standard bundle-size, 5 is chosen as the standard bundle-size together with a sloppy way of writing numbers hiding both the decimal point and the unit so that e.g. 3.2 5s becomes first 3.2 and then 32, thus introducing place values where the left 3 means 5-bundles and the right 2 means unbundled singles. This leads to the observation that the chosen bundle-size does not need an icon since it is never used when using place values, or in the counting sequence: 1, 2, 3, 4, bundle, 1B1, 1B2, 1B3, 1B4, 2B, 2B1, etc. So once chosen, ten needs no icon.

Likewise, in the beginning, counting in tens should use neither a ten-icon nor the ten-name, but count 8, 9, bundle, 1bundle1, 1B2, 1B3, …, 1B9, 2B, 2B1, etc. Then the name bundle can be replaced by the name ten counting 8, 9, ten, 1ten1, 1T2, …, 1T9, 2T, 2T1, etc. Finally the sloppy way eleven and twelve can be used meaning '1 left' and '2 left' in 'Anglish', i.e. in old English.

Finally Introducing 10

Introducing the number ten as the standard bundle-size changes almost everything.

Numbers are no more written as natural numbers, i.e. as decimals carrying units. Instead numbers are written using the sloppy place-value method hiding both the decimal and the unit.

Since ten has no icon, double-counting now becomes impossible to predict by formulas since asking 8 3s = ? tens leads to T = (8*3/ten)*ten that cannot be calculated. Now the answer is given by multiplication, 8*3 = 24 = 2 tens + 4 ones, thus transforming multiplication into division.

Almost all operations change meanings. *3 now means recounted in tens. /4 now means divided in 4, not divided into 4s. +3 now means adding 3 on top, not next to. Only –3 still means take away 3.

With 2.order counting in e.g. 5s the order of operations is: first /, then *, then –, and finally +. With 3.order counting in tens this order is turned around: first +, then –, then *, and finally /.

The Natural Roots of Middle School Mathematics

Also in middle school, the natural root of mathematics is double-counting the physical fact many, now occurring different places in the outside world, especially in time and space and in economy, thus reintroducing units, now as e.g. seconds, minutes, cm, m, m^2, m^3, liters, c, $, £ etc.

Double-Counting in Middle School

In middle school, double-counting means counting a given object in different units. Fractions emerge when counting 3 1s in 5s as 3 = (3/5)*5; and as ‘per-numbers’ when a quantity is counted e.g. both as 2$ and as 5kg thus containing 2$ per 5kg, or 2$/5kg or 2/5 $/kg. Again the recount-formula predicts recounting results when asking ‘6$ = ?kg’, or ‘14kg = ?$:

T = 6$ = (6/2) * 2$ = (6/2) * 5kg = 15kg; and T = 14kg = (14/5) * 5kg = (14/5) * 2$ = 5.6$, or

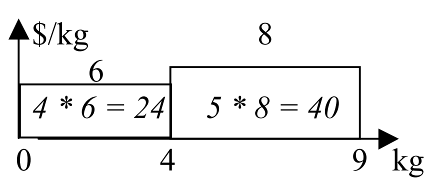
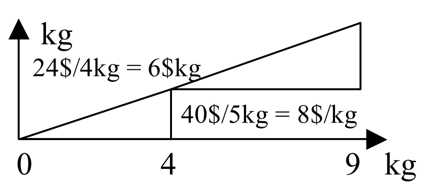
kg = kg/$ * $ = 5/2 * 6 = 15 , and $ = $/kg * kg = 2/5 * 14 = 5.6.

Percentages are per-numbers using the same unit. Thus the females might contribute with 3 females per 8 persons, or 3 females/8 persons, or 3/8 f/p, or 3/8 if we leave out the unit. Thus per hundred, the females will contribute with T = 100p = (100/8) * 8p = (100/8) * 3f = 37.5f, i.e. with 37.5 females per 100 persons, or 37.5 females /100persons, or 37.5/100 f/p, or 37.5/100, or 37.5%.

Thus percentages always carry a hidden unit. So when finding the number of females in a population of 75 having 20% females, the hidden unit is reintroduced: With 20% females we have 20 females per 100 persons, so the 75 persons must be recounted in 100s:

75 persons = (75/100) * 100 persons = (75/100) * 20 females = 15 females

Primary school integration adding stacks by adding bundles, e.g. 2 3s + 4 5s = ? 8s, now reoccurs as adding per-numbers when mixing two double-counted quantities, asking e.g. 4 kg at 6$kg + 5kg at 8$/kg = 9 kg at ? $/kg. This question can be answered by using a table or a graph.

4kg at 6$/kg = 4 * 6 = 24 5kg at 8$/kg = 5 * 8 = 40		
9kg at ?$/kg = 9 * ? = 64 ? = 64/9		

Using a graph we see that integration leads to finding the area under a per-number graph; and opposite that the per-number is found as the gradient on the total-graph.

The Root of Equations: Reversed Calculations

In Greek, mathematics means knowledge, i.e. what can be used to predict with. So mathematics is our language for number-prediction: The calculation ‘2+3 = 5’ predicts that counting on 3 times from 2 will give 5. ‘2*3 = 6’ predicts that repeating adding 2 3 times will give 6. ‘2^3 = 8’ predicts that repeating multiplying with 2 3 times will give 8. Also, any calculation can be turned around and become a reversed calculation predicted by the reversed operation:

The answer to the reversed calculation 3 + x = 7 is predicted by the reversed operation x = 7 –3.

The answer to the reversed calculation 3 * x = 7 is predicted by the reversed operation x = 7/3.

The answer to the reversed calculation x ^ 3 = 7 is predicted by the reversed operation x = $\sqrt[3]{7}$.

The answer to the reversed calculation 3 ^ x = 7 is predicted by the reversed operation x = $\log_3(7)$.

Thus the natural way to solve an equation is to move a number across the equation sign from the left forward-calculation to the right backward-calculation side, reversing its calculation sign:

3 + x = 7 x = 7 – 3	3 * x = 7 x = 7/3	x^3 = 7 x = $\sqrt[3]{7}$	3^x = 7 x = $\log_3(7)$

A calculation with more than one operation contains an invisible bracket placed around the calculation having priority, and disappearing again with the number outside the invisible bracket:

	2*x + 3 = 11	(2*x) + 3 = 11	2*x = 11-3	x = (11-3)/2
Or with letters:	a*x + b = c	(a*x) + b = c	a*x = c – b	x = (c – b)/a

The Root of Geometry: Triangulation

In Greek, geometry means earth-measuring. Earth is measured by being divided into triangles; which again can be divided into right-angled triangles, each seen as a rectangle halved by a diagonal. Recounting the height and base produces per-numbers: sinA = height/diagonal, tanA = height/base, cosA = base/diagonal. Additional formulas are A+B+C = 180, and a^2+b^2 = c^2.

Also a circle can be divided into many right-angled triangles whose heights add up to the circumference C of the circle: C = 2 * r * (n * sin(180/n)) = 2 * r * π for n sufficiently big.

The Roots of Algebra: Reuniting

In Arabic, algebra means reuniting, i.e. splitting a total in parts and (re)uniting parts into a total. The operations + and * unite variable and constant unit-numbers; ∫ and ^ unite variable and constant per-numbers. The inverse operations – and / split a total into variable and constant unit-numbers; d/dx and √ and log split a total into variable and constant per-numbers:

Totals unite/split into	Variable	Constant
Unit-numbers $, m, s, ...	T = a + n $T - n = a$	T = a * n **Error!** = a
Per-numbers $/m, m/s, m/100m = %, ...	ΔT = ∫ f dx **Error!** = f	T = a ^ n $\sqrt{n;T} = a$ $log_a T = n$

The Natural Roots of High School Mathematics

In high school, the natural root of mathematics is double-counting change: Being related by a formula y = f(x), how will a change in x, Δx, affect the change of y, Δy? Here the recount-equation gives the change-formula Δy = (Δy/Δx)*Δx, or dy = (dy/dx)*dx = y'dx for small micro-changes.

Constant Change

In trade, when the volume increases from 0 to x kg, the initial cost b increases to the final cost y = b + a*x if the cost increases a$/kg. This is called linear change or ++ change. Here Δy/Δx = a.

In a bank, when the years increase from 0 to x, the initial capital b increases to the final capital y = b*(1+r)^x if the capital increases with r %/year. This is called exponential change or +* change since we add 7% by multiplying with 107% = 1 + 7%. Here Δy/Δx = r*y.

In geometry, when the side-length is 3-doubled from 2 to 6, the area of a square y = x^2 is 3-doubled twice from 4 to 36. This is called potential change or ** change, in general given as y = b*x^a. Here Δy/Δx = a * y/x.

In a linear change formula y = b + a*x, the per-number a might itself change in a linear way as a = c + d*x. In this case we have y = b + a*x = b + (c + d*x)*x = b + c*x + d*x^2, a polynomial of degree 2. Here Δy'/Δx = d, where y' = Δy/Δx. If also d changes in a linear way as d = m + n*x then y becomes a polynomial of degree 3, y = b + c*x + d*x^2 = b + c*x + (m + n*x)*x^2 = b + c*x + m*x^2 + n*x^3. Here Δy''/Δx = n, with y'' = Δy'/Δx and y' = Δy/Δx.

Thus in y = a + b*x + c*x^2 + d*x^3, 'a' describes the initial height; 'b' the initial change or steepness; 'c' the acceleration or the curvature; and 'd' the counter-curvature. Thus the y-graph becomes a line bending first to one side, then to the other, i.e. a double-parabola.

Predictable Change

The change in x, Δx, might be a micro-change, dx. On a calculator we observe that, approximately, 1.001^5 = 1.005, 1.001^9 = 1.009, √1.001 = 1.0005 and 1.001^-3 = 0.997. From this a hypothesis can be made saying that, approximately, (1+dx)^n = 1 + ndx, allowing numerous predictions to be made as 1.0002^4 = 1.0008 etc., all being verified by the calculator. If y = x^n, then (x + dx)^n = (x(1+dx/x))^n = x^n(1+ndx/x) = x^n + ndx*x^n/n = y + ndx*x^(n-1) = y+dy, so dy/dx = n*x^(n-1).

If the changes in x, Δx, are micro-changes, dx, the area under a graph y = f(x) can be found by summing up the micro-strips under the graph, each having the area height*width = f*dx, written as **Error!**. If y dx can be written as a micro-change df, then **Error!** = **Error!** = f(b) – f(a) since summing up single changes always gives a total change = terminal number – initial number, no matter the

size of the single changes. So with x^3 = (**Error!**)', **Error!**= **Error!**= **Error!**– **Error!** = 152.25.

Change Equations

Solving any change-equation dy/dx = f(x,y) is easy when using technology. In such equations, the change-equation calculates the change dy that added to the initial value gives the terminal y-value, becoming the initial value in the next period. Thus the change equation is dy = r*y, r = ro(1-y/M) if a population y grows with a rate r decreasing in a linear way with the population y having M as its maximum. This is easily solved using a spreadsheet to keep on calculating the formula y+dy -> y.

Unpredictable Change

From throwing a dice we know that some numbers change in a way that cannot be 'pre-dicted'. However, such numbers x can be 'post-dicted' by a table describing their previous behaviour with relative frequencies f. From such a table we can calculate the average level m = Σ(x*f) and the average deviation d from the average level, d^2 = Σ((x – m)^2*f). This allows for predicting new numbers, not by a value, but by an interval, m ± 2d, containing the numbers with a 95% probability. Likewise, ordering the observed numbers and splitting them in four parts will produce a box-plot.

Relating Equations and Curves

Placed in a XY coordinate-system, the points of a curve are assigned coordinates (x,y). This allows a curve to be translated into an equation and vice versa. So now curve-problems can be translated into solving equations, and vice versa. Intersection points, turning points, tangents, parallelism, areas etc. now can be translated into solving simultaneous equations, differentiating and integrating formulas; and integrals is found as areas etc. The nature of formulas as means for prediction can be illustrated by using the geometry solution as a prediction of the algebra solution, and vice versa.

Using regression, a graphical calculator can translate any table into a formula and a curve.

Modelling With Regression-Mathematics

Modelling consists of four parts: a real-world problem, a model problem, a model solution, and a real-world solution. First a real-world problem leads to a model problem, often a table relating two variables x and y. Then in the model solution, regression is used to find a formula connecting the variables. Containing one unknown, the formula becomes an equation that can be solved manually or by the Math Solver. Containing two unknowns, the formula becomes a function, that can be illustrated by a graph; and where the two typical questions 'given x find y' and 'given y find x' reduces the function to an equation that can be solved manually or by the Math Solver, or by the Trace and the Calc Intersection. Using both provides the opportunity to use the first as a prediction of the second. Finally the solution can be evaluated as to its applicability as a real-world solution.

The Three Genres of Modelling

A formula can predict. However, in the formula T = c*p we need to know what quantities are described to determine the truth-value of the formula's prediction. It turns out that both word-statements and number-statements share the same genres: fact, fiction and fiddle (Tarp, 2001).

Fact Models

Fact models quantify and predict predictable quantities: 'What is the area of the walls in this room?' In fact models the predicted answer is what is observed. Hence calculated values from a fact models can be trusted. The basic formulas T = 2*3 etc. are fact models, as well as many models from basic science and economy. A fact model may also be called a 'since-hence' model or a 'room-model'.

Fiction Models

Fiction models quantify and predict unpredictable quantities: ‘My debt will soon be paid off at this rate!’ Fiction models produce predictions based upon presumed assumptions that should be supplemented with alternative assumptions, i.e. with parallel scenarios. Typical examples of fiction models are average-models, simplifying complex economical or technical models by assuming some variables to stay as constants on their average level. Other examples are linear demand and supply curves in economical theory. A fiction model may also called an ‘if-then’ or a ‘rate-model’.

Fiddle Models

Fiddle models quantify qualities: ‘Are the risk and casualty numbers of this road high enough to cost a bridge?’ This question will install crosswalks instead of bridges on motorways since it is cheaper to be in a cemetery than at a hospital. Fiddle models should be rejected asking for a word description instead. Many risk-models are fiddle models. The basic risk model says: Risk = Consequence * Probability. A fiddle model may also be called a ‘so-what’ model or a ‘risk-model’.

The Grand Narratives of the Quantitative Literature

Literature is narratives about real-world persons, actions and phenomena. Quantitative literature also has its grand narratives. That overwhelmingly many numbers can be added by one simple difference, providing the numbers can be written as change-numbers, is a grand narrative.

In physics, grand narratives can be found among those telling about the effect of forces, e.g. gravity, producing parabola orbits on earth, and circular and ellipse orbits in space. Jumping from a swing is a simple example of a complicated model. These grand narratives of physics enabled the rise of the Enlightenment period and the modern democratic society replacing religion with science.

In economics, an example of a grand narrative is Malthus’ ‘principle of population’ comparing the linear growth of food production with the exponential growth of the population; and the Keynes model relating demand and employment creating the modern welfare society. As are the macroeconomic models predicting the effects of different taxation and reallocation policies.

Also limit-to-growth models constitute grand narratives predicting the global economical and ecological future depending on different production, consumption, and pollution options.

Pastoral vs. Grounded Mathematics in Primary School

In primary school, 6 is introduced as a symbol for six being the follower of five, having the symbol 5. 10 is introduced as a symbol for ten, the follower of nine having the symbol 9. And the counting-numbers 1, 2, 3 etc. are presented as the natural numbers. The hidden grounded alternative to these pastoral choices introduces 5 and 6 as what they really are: icons rearranging the sticks they represent; and 10 as what it really is: a sloppy way of writing 1.0 bundle, thus being the follower to 4 in the case of 5-bundling making the follower of nine 20; and the natural numbers as what they really are: decimal-numbers with units, e.g. 0.4, 1.0, 1.1, …, 1.4, 2.0 when counting in 5-bundles.

The tradition introduces division as the last of the four operations, where /4 means to split in 4. The hidden grounded alternative introduces division as what it really is: an icon for taking away splitting in 4s, where 3*7/4 predicts how many times 4s can be taken away from 3 7s.

The tradition introduces multiplication as the third of the four basic operations, where the multiplication tables, 3*8 = 24 etc., forces all stacks to be recounted in tens. The hidden grounded alternative introduces multiplication as what it really is: 3*8 means a stack of 3 8s, not needing to be recounted into tens before after ten has been chosen as the standard bundle-size.

The tradition introduces addition as the first of the four operations, where e.g. 7 + 4 = 11 forces the immediate introduction of ten as the bundle-size; and forces the sloppy way of writing 2digit numbers without decimals and units. The hidden grounded alternative introduces addition as what it really is: 2 5s + 4 1s means juxtaposing a stack of 4 1s next to a stack of 2 5s.

The tradition introduces 'mathematism', true in the library but not in the laboratory, by teaching that '2 + 3 IS 5' in spite of the fact that 2weeks + 3days = 17days, 2m + 3c = 203cm etc. The hidden grounded alternative always includes the units when adding, e.g. 2 4s + 3 5s = 4.3 5s.

Pastoral vs. Grounded Mathematics in Middle School

Middle school introduces fractions as 'rational' numbers having decimals and percentages as examples; and allowed to be added without units. The hidden grounded alternative to this pastoral choice introduces decimals and fractions as what they really are: decimals occur as the natural numbers when counting in bundles; and fractions are per-numbers occurring when double-counting a quantity in two different units, as 1s and as 5s: 3*1 = (3/5)*5; and as $ and kg: 2$/5kg = 2/5 $/kg.

Factorization is introduced to create a common denominator when adding fractions, and for solving quadratic equations. The hidden grounded alternative includes units when adding fractions, and postpones quadratics to high school. Factorization unfolds folding numbers to prime numbers.

Equations are introduced as equating 2 number-names to be changed by identical operations aiming at neutralizing the numbers next to the unknown. The neutralizing method seems to have as a hidden agenda to legitimize the concepts of modern abstract algebra in teacher education solving a simple equation as 3 + x = 8 by using, not the definition of reversed operations, but both the commutative and associative laws, and the concepts of inverse and neutral elements of number sets: 3 + x = 8 , (3 + x) + (-3) = 8 + (-3) , (x + 3) + (-3) = 8 – 3 , x + (3 + (-3)) = 5 , x + 0 = 5 , x = 5.

The hidden grounded alternative to this pastoral choice introduces equations as what they really are: calculations being reversed since we know the result, but not the starting number.

Multiplying two digit brackets is introduced complaining about it is not possible to explain why (–1) * (–1) = +1. In a grounded approach this is no problem since the multiplication 9*9 = 81 implies that (10-1)(10-1) = 100 – 10 – 10 + (–1) * (–1), clearly showing that (–1) * (–1) = +1.

Geometry is introduced as forms and facts deduced from self-evident axioms. The hidden grounded alternative to this pastoral choice introduces geometry as what it really is: 'earth-measuring' realising that all forms can be split into right-angled triangles, where the relationship between the angle and the side can be expressed by the percentage numbers sinA, cosA and tanA.

Pastoral vs. Grounded Mathematics in High School

High school introduces formulas as set-relations: polynomial, exponential, and circular functions. The hidden grounded alternative introduces these formulas as what they really are: solutions to change-equations rooted in describing physical motion and population and economical growth.

Calculus is introduced as an example of a limit process, thus introducing limits and continuity before the derivative. The hidden grounded alternative generalizes primary school's adding stacks in combined bundle-sizes, and middle school's adding fractions with units into adding per-numbers with units; and introduces the terms continuous and differentiable as what they really are: foreign words for locally constant and locally linear in contrast to piecewise constant and piecewise linear.

Conclusion

Three questions have been answered using anti-pastoral sophist-research. In primary, middle and high school the core mathematical concepts are not grounded in nature. Replacing ungrounded concepts with grounded concepts constitutes a deconstruction of a self-referring mathematics curriculum into a grounded curriculum. This allows mathematics education to change from being pastoral patronization to democratic enlightenment. Now only the political decision remains. Is mathematics education meant to demonstrate how real-world phenomena are examples of metaphysical forms only visible to mathematicians training teachers to mediate this insight to the ordinary people? Or is mathematics education meant to demonstrate how mathematical concepts are rooted in and able to predict the behavior of real-world phenomena? In short, should mathematics education patronize the ignorant to become a docile lackey – or should mathematics education enlighten nature so that people can tell nature from choice and practice democracy?

References

Biehler R., Scholz R.W., Strässer R. & Winkelmann B. (1994). *Didactics of Mathematics as a Scientific Discipline*. Dordrecht: Kluwer Academic Press.

Denzin N. & Lincoln Y. (2000). *Handbook of Qualitative Research 2nd ed.* London: Sage.

Glaser B. G. & Strauss A. L. (1967). *The Discovery of Grounded Theory.* NY: Aldine de Gruyter.

Jensen J. H, Niss M. & Wedege T. (1998). *Justification and Enrolment Problems in Education Involving Mathematics or Physics*. Roskilde: Roskilde University Press.

Kline M. (1972). *Mathematical Thoughts from Ancient to Modern Times*. NY: Oxford U.P.

NCTM (2000). *Principles and Standards for School Mathematics*. National Council of Teachers of Mathematics, Reston VA.

Russell B. (1945). *A History of Western Philosophy.* New York: A Touchstone Book.

Tarp A. (2001). Fact, Fiction, Fiddle - Three Types of Models. in J. F. Matos & W. Blum & K. Houston & S. P. Carreira (Eds.). *Modelling and Mathematics Education, ICTMA 9: Applications in Science and Technology*. (pp. 62-71). Chichester UK: Horwood Publishing.

Tarp A. (2004). *Pastoral Power in Mathematics Education.* Paper accepted for presentation at Topic Study Group 25. The 10th Int. Conf. on Mathematics Education, ICME 10, 2004.

Tarp A. (2008). *CATS, Count&Add in Time&Space - a Natural Way to Become a Mathematics Teacher.* Paper accepted at Topic Study-group 27. ICME 11, 2008. www.MATHeCADEMY.net.

Tarp A. (2009). *Double-Counting.* An extended version of this paper. www.MATHeCADEMY.net.

Zybartas S. & Tarp A. (2005). One Digit Mathematics. *Pedagogika (78/2005)*. Vilnius, Lithuania.