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WHAT CAN WE LEARN FROM LOGICAL ANALYSIS OF MATHEMATICAL TASKS WITH A SEMANTIC PERSPECTIVE?

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Introduction and theoretical frame

Quine (1987) claims that

> "Despite [some] exclusion, all the austere science submits pliantly to the Procrustean bed of predicate logic. Regimentation to fit it thus serves not only to facilitate logical inference, but to attest logical clarity" (op.cit.; 158)

All his life, Quine supported the thesis of the fruitfulness of logic in analysing scientific discourse and reasoning as well as ordinary language. This position seems contradictory with a general assumption that logic is of no use for mathematics (Dieudonné,1987; Thurston, 1994), or that logical deduction is necessarily sterile (does not product new truths). However the focus of Quine is clearly on a semantic perspective of truth, as developed by Tarski (1933; 1936), relying on objects, properties, relationships, variables, connectors and quantifiers and the central notion of satisfaction of a formulae by elements of a relevant interpretation. All of these logical concepts are at the very core of mathematical activity, and one of the main interests of the semantic point of view in logic is to propose a formalisation in which the power of syntax methods leads to results that can be applied in various interpretations, providing a genuine articulation between *form and content* (Sinaceur, 1991 ; 2001). From this point of view, logical analysis contributes to the enlightenment of the meaning of statements and to control the validity of proofs. In my own research in mathematics education, I have shown the fruitfulness of this logical frame for enriching the *a priori* analysis of mathematical tasks and for analysing and interpreting students' argumentation and proofs. This provides tools to look at students' reasoning abilities in a more positive way than is generally done from a syntactic perspective (Durand-Guerrier, 1995; 2003a; 2003b; 2005). One of the main questions faced by mathematical educators concerns the type of task that can be proposed to students in order to allow them to gasp mathematical concepts and to deal both with formal and non-formal aspects of these concepts. Following Tarski's methodology of deductive science, we make the hypothesis that axiomatic systems and theory emerge from action, or more precisely emerge in a *back and forth* between active work with familiar mathematical objects and theoretical elaboration. In this perspective, it is possible to build new abstract mathematical objects from familiar mathematical objects, that play the role of concrete objects, in the sense that *« Le concret, c'est de l'abstrait rendu familier par l'usage »*[1] (Langevin, 1950). In a didactic perspective, that means that actions play a very fundamental role in conceptualisation, so that an aim for researchers and educators is to look for mathematical situations that can offer didactic opportunity for students to build authentic mathematical knowledge, beyond the mastery of mathematical language. This project is supported by Guy Brousseau (1998), whose creativity for such situations is attested in various mathematical fields. A famous example is given by "the enlargement of a puzzle" which

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[1] *"Concrete objects are abstract objects that became familiar through their use" (our translation)*

appears as a powerful situation to give a genuine content to the formal rule "to enlarge a geometrical figure in respect with its form, it is necessary to multiply its size". Indeed, the more spontaneous proposal from ten or eleven ages is that to enlarge a figure so that a segment with measure of length 4 cm became a segment with measure of length 7 cm, it is to add 3 cm to each size. This situation is part of a proposition for teaching decimal numbers at elementary school. As shown by Bronner (1997), the specificity of decimal numbers is insufficiently developed in French curriculum; as a consequence, students are mainly unable to make a clear distinction between decimal numbers and "idecimal" numbers[2], which is not the case for irrational numbers that are explicitly studied in secondary school. To find a relevant task that deals authentically with this question is an interesting challenge, due in particular (but not only) to the fact that logical matters are here strongly intertwined with the mathematical nature of objects. I will present now a task that, according to me, enlightens meaning for students through a *back and forth* between active work with familiar mathematical objects and theoretical elaboration.

An example of a task intertwining logical and mathematical aspects

For this presentation, I have chosen a mathematical task proposed to French upper secondary school students (grade 16) in a specific solving problem frame[iv]. Context, task and students' activity are described and analysed in Pontille & al. (1996). At the very beginning of the school year, a mathematician proposed in two high schools in Lyon some research problems; volunteer students chose one problem, and worked on it in a small group all year. They were accompanied by their teachers and by the mathematicians. The authors of the papers have observed them; they made some audio and video recording, student interviews, and collected writings.
The problem is the following one:

> "Given a function f between {1, 2,…*n*} and {1, 2, …*n*}, with *n* an integer different from zero. *f* is supposed to be increasing; show that exists an integer *k* such that *f*(*k*) = *k* ; *k* is named fixed point. Study possible generalisations in the following cases, with *f* increasing:
> $f : D\cap[0, 1] \rightarrow D\cap[0, 1]$ D is the set of decimal numbers
> $f : Q\cap[0, 1] \rightarrow Q\cap[0, 1]$ Q is the set of rational numbers
> $f : [0, 1] \rightarrow [0, 1]$
> or any other generalisation.

The students were from two different high schools; they met four times during the year, and at spring a small congress was organized where students gave a poster and an oral presentation. Throughout their work, they wrote down their results, questions, conjectures, proofs, in a notebook.
We present aspects of the logical and mathematical analyses and some insights about the students' work on the two first subtasks.

The first subtask with integers
This task has some important characteristic features that I will shortly present. First of all, it is easy for students to understand the question, and to deal with it ; indeed, at that level, integers are really familiar. To enter the problem, two main paths could be taken: the first one consists

[2] The word « idecimal » was introduced by Bronner (1997) to describe numbers that are not decimal, that means with a non-finite decimal development.

of trying a classical function restricted to the set of integers; the second one is to give small values to *n*, and to observe what occurs, then the conjecture that the sentence is true can emerge, along with an argument for the proof. The first approach can reinforce the conviction that it is true, but does not indicate idea for proof. The proof has to be done for a generic function, and the conclusive argument leans on the fact that the set is discrete and bounded, and therefore finite. Dealing with small values for *n* may suggest the reason why it is impossible that for all *k* between 1 and *n*, *f*(*k*) is different from *k* : under such an hypothesis, as *f* is increasing, it would be necessary that *f*(*n*) were always greater than *n* - a contradiction with the given. After four weeks, the students wrote a proof with reduction *ad absurdum* relying on these arguments.

At the beginning of the proof they wrote: "reformulation de l'énoncé: Montrer que n'importe qu'elle fonction *f* correspondant aux conditions de l'énoncé possède au moins un point fixe" (a new formulation for the question: Show that for any function *f* satisfying the given conditions has at least one fixed point). We can see, in this reformulation, a rigorous expression of the logical nature of the question that leads to a proof by generic element. This first subtask appears as a preliminary that allows students to understand well that the question is a very general one: 'what does it mean for a function to have a fixed point?', and to elaborate a first schema of proof.

A first generalisation to work on idecimality

Once students have got a proof for the case of integers, they are invited to work with the possible generalisation, the first question concerning the set of decimal numbers, then the set of rational numbers and finally the set of real numbers. As for the previous subtask, the logical structure of the statement is of type "for all x, there exist y"; in the considered interpretations, variable *x* takes its value among functions, while, given a function, *y* takes its value in the domain of the function. Due to the similarity of the logical structure, it is likely that a similar schema of proof will first appear. However, it is clear that the argument provided for integers could no longer be used. Changing the domain of the function involves here a change in the objects themselves. The main difference is that for integer, every number has a successor, that not hold for decimal numbers, for which, between every pair a decimal, it is always possible to put a decimal, different from the two previous ones. This situation provokes the encounter, by students, of this strong difference between decimal and integers, and this is what happened for the students that were observed. Indeed, as could be anticipated, as a first step students typically try to transfer to the set of decimal numbers what worked so well for integers. While they are doing this, many important epistemological questions arise: is it possible to consider that there is a fixed distance between two decimals; is it possible to find a smaller positive decimal?; are these two questions related or independent? The students finally consider that there is not a unique 'smaller' positive decimal, and finally that there does not have to be a fixed distance between two decimals.

To continue the research, the students are faced to the following problem: is it possible that the graph of an increasing function for which both the domain and co-domain are the set of decimal numbers D restricted to the interval [0, 1] intersects the first bisector outside of D? At this point of the research, the importance of the domain of quantification appears anew clearly: searching a solution in D or in Q, or searching a solution in R, are very different mathematical questions, and this is in spite of the three subtasks seeming to be identical with just a difference of the domain. It is an important feature of this task that the change in the domain might change the answer to the question. Indeed, a common practice in mathematics is to remain implicit the domain of quantification, considering generally in such a context that this domain is the real numbers set. Doing this, the specificity of real numbers is hidden, and the power of visual evidence is reinforced. This relates to the completeness of R, in accord with the graphical evidence, and the incompleteness of D and Q, which are difficult to perceive. Indeed, we know well how to represent a discrete set (N), or a continuous set (R); but providing a representation of those sets which, as with D and Q, are countable and non discrete, is very difficult, and hence, first perception is of little use. To deal with this question, students provide various "discrete" representations of decimal numbers: alternation of decimal and idecimal numbers; representation with bicoloured squares; continuous straight line with indication of holes. The importance of this last representation appears in the oral exchanges between the students and seems to play an important role in the path to a solution: "we have seen that in the decimal line, there are holes"; "in fact, we will have a dotted line with sometimes decimal numbers, sometimes rational numbers, sometimes real numbers completely intermingled, so that we have to link together our points on a line, if not we can't do anything". This leads to the following question, written on the notebook: "is it possible that the graph of the function *f* cross the graph of "y = x" without going through a point of "y = x" with decimal coordinates?". After many exchanges, they conclude that it is possible that the crossing is on a hole. Turning back to the initial question, they look for a counterexample. One of the students provides a counter example with an affine function (a rational idecimal number), and one with a quadratic function (a real, irrational and idecimal number), precising that they are looking for a counterexample for $f : D \cap [0, 1] \to D \cap [0, 1]$ with simple concrete functions. It took about five months from the beginning till students provide this solution that gives also a solution for the case where the domain for the function is the rational number set.

Conclusion

In this paper, I intend to argue that the semantic perspective of this task opens interesting and rich opportunities for authentic mathematical work with students. In the task I have presented the syntactical logical structure remains fixed throughout the task, while the interpretations of the task in the various sets of functions, characterized by the domain of the function, change the possible mode of reasoning and necessitate a deep understanding of the nature of the numbers involved in the subtask. Our description of the students' work shows that students develop both reasoning ability and mathematical knowledge. It also shows that looking for a counterexample appears only after long work reconsidering, reorganizing, and adapting their knowledge of decimal numbers. Although the context is far from ordinary teaching, according to me this example supports the hypothesis that it is valuable to propose to students such types of task, allowing work *back and forth* between formal and informal aspects, in order to provide a deep understanding of mathematical objects through significant reasoning activity.

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