Measurement and Multiplicative Thinking in Prospective Elementary Teachers

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Abstract

Measurement and units are the foundation for multiplicative thinking, which is in turn the basis of most secondary mathematics. During the 20th century, curriculum in U.S. schools has been gradually become almost entirely focused on syntax and algebra. This paper reports on preservice elementary teachers' performance on a set of simple questions involving measurement and multiplicative thinking. We conclude that lack of opportunity to learn measurement skills is at the base of most of the problems in a deep understanding of elementary mathematics. We propose a partial solution in the form of a textbook-in-progress for preservice elementary teachers. This textbook is founded on measurement rather than set theory and is heavily activity-based.

Measurement and multiplicative thinking

The syntax of the real numbers was developed to model measurement (including counting, which measures the size of finite sets.) It is not a coincidence that addition gives the combined length of two joined line segments, or that that the area of a rectangle is the product of its length and width. If it did not, there would have to be a parallel system of mathematics modeling measurement.

Measurement, at its most elementary level, is a comparison of a quantity of two objects. Quantities are attributes that can be quantified: for example, length, area, volume, weight, the size of a finite set, time, money, etc. With the choice of a unit, such as a meter, an inch, or the length of one's foot, numbers arise by comparing the given quantity with copies of the unit. Thus a measured quantity consists of a number and a unit. The unit is arbitrary, and the number assigned to a given quantity changes as the unit changes. In modern mathematics, numbers tend to be abstract, unitless objects. However, in any use of numbers for a practical purpose (aside from identification numbers), numbers always have a unit attached.

Number lines are one of the most important ways of representing real numbers in modern mathematics. A number line gives a length model for number, or for whatever quantity is being represented. A ruler is an example of a number line, as is a linear scale on an analog measuring instrument: a graduated cylinder, a spring scale, a thermometer. Perpendicular number lines define the Cartesian plane, which has become so central to our mathematical consciousness that is difficult to conceive of a space without a coordinate system.

It has been well established (Vergnaud (1994), Lamon (2005)) that the main concepts in the complex known as multiplicative thinking are the processes of unitizing: choosing an appropriate unit for the problem at hand, and reunitizing: changing units flexibly as necessary. For example, a common fraction is a number with an implicit unit (1), a fractional unit, described by the denominator, and a number describing the number of copies of the fractional unit (the numerator.) Adding fractions requires several units to be held simultaneously:

$\frac{2}{3}+\frac{1}{2}$	Units: wholes, thirds, halves
$=\frac{4}{6}+\frac{3}{6}$	Change wholes and halves to sixths (new unit)
$=\frac{7}{6}$	Combine like units
$=1\frac{1}{6}$	Change 6 sixths to one whole

Unitizing is also the basis for working with ratios. For instance, if the lengths of two pencils are 4 inches and 6 inches, then the longer pencil is 3/2 of a short pencil length, and the short pencil is 2/3 of a long pencil length.

Multiplication as scaling is the basis of geometric similarity as well as for scalar multiplication of quantities that are essentially vectors, such as recipes. Repeated scaling is the basis for place value and for exponential and logarithmic functions. The necessity of multiplicative thinking in understanding secondary mathematics was briefly outlined in Confrey (1994). Her research shows that logarithms can be taught effectively to students who can think multiplicatively; college students who don’t understand logarithms usually also are weak in multiplicative thinking.

Research results on preservice elementary teachers.

The instrument. The questions described below were asked to students at a commuter university in a working class, urban/suburban area in California. A majority of students at this university are the first generation in their family to attend college. The main ethnic subgroups in 2006 were 37% Hispanic, 36% white, and 13% African American.

The questions were given to students in three courses: a "general education" first year math course (Math 115), at the beginning of the first course for elementary teachers (Math 301A), and at the beginning of a capstone problem solving course for elementary teachers (Math 308); students in this course had completed a year of math courses for elementary teachers. We believe these students are fairly typical of preservice elementary teachers in California.

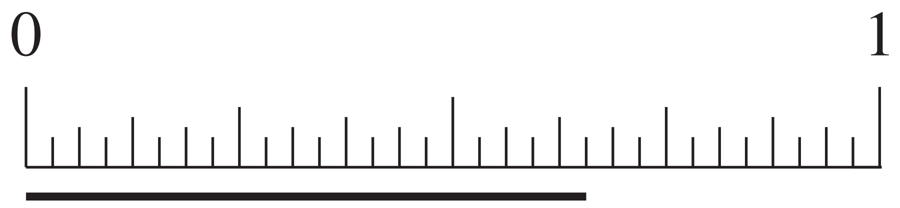
There was no significant difference in the responses between the preservice teachers in Math 115 and Math 301A, or between preservice teachers and non-teachers in Math 115. There were significant differences between the Math 115 and Math 301A students ("before" in the tables below; N=89) and the Math 308 students ("after", N=25).

The questions were given as a take-home extra-credit quiz, with credit given for completing the quiz regardless of performance. The questions presented here were selected for this report because they directly use measurement and/or units, and because significant numbers of students could not answer them correctly.

Questions and discussion. Many of the questions were formulated with customary U.S. units rather than metric units because students tend to be more comfortable with them.

The first two problems involve common and decimal fractions.

1. In the imaginary country of Elbonia, the basic unit of length is the elbo. Here is a piece of a ruler in elbos. How long (in elbos) is the heavy line segment beneath the ruler?



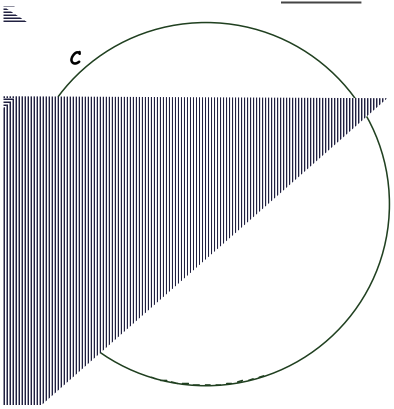
This question is included because the incorrect answers in the "before" group reveal student thinking, as confirmed by selected interviews. Most errors were unit errors; for instance, a response of $2\frac{5}{8}$, instead of the correct length of $\frac{21}{32}$ elbos, often meant that the student used an inch as the unit. (One fourth of an elbo is close to an inch on the test paper.) An answer of $5\frac{1}{4}$ or $10\frac{1}{2}$ often meant a centimeter as the unit. Some explicitly stated a standard unit, such as 21 mm. A more serious error assumed that the ruler was marked in a decimal system, giving a response of .21. Fortunately, a large majority of “after” students could do this problem.

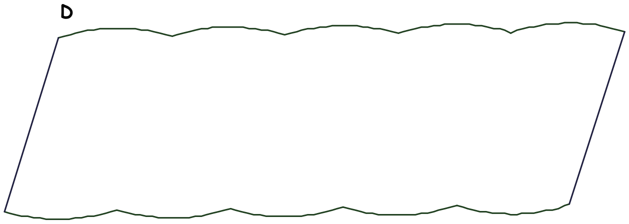
The next four problems concern length, area, and volume, the first in a qualitative way, and the other three quantitative. We assumed that students knew formulas for the area of a rectangle and volume of a rectangular prism, but we provided a formula for the area of a parallelogram.

Elbo Ruler

	before	after
Correct or approx. correct	55%	84%
Unit error	33%	8%
Other error	8%	4%
Not done	4%	4%
Total	100%	100%

2. Don’t use any measuring devices: just think. Fill in the blanks with <, =, or >. Briefly explain your reasoning for both answers.
   a. Area of C _____ Area of D
   b. Perimeter of C _____ Perimeter of D





Circle Dissection Area

	before	after
Correct w. explanation	38%	52%
Correct w/o explanation	45%	48%
Incorrect	16%	0%
Not done	1%	0%
Total	100%	100%

Circle Dissection Perimeter

	before	after
Correct w. explanation	15%	44%
Correct w/o explanation	37%	24%
Incorrect	48%	32%
Not done	0%	0%
Total	100%	100%

The area question was easier; even if a student is not entirely sure what is meant by area, the two shapes clearly have the same amount of stuff. Perimeter requires a more subtle analysis, but the most prevalent incorrect answer (by a ratio of 3 to 1) was that the perimeters are also equal. Students giving this answer will often say that all measurable attributes of the two figures are the same.

3. How many square inches are in a square foot?

Virtually all students know that there are 12 inches in a foot, but most also think that there are 12 square inches in a square foot. Some answered 48 square inches, because a square has four sides. Evidently 40% of students, even after a year of mathematics for elementary teachers, do not really understand what square units are.

Square Inches in a Square Foot

	before	after
Correct	33%	60%
Incorrect	58%	36%
Not done	9%	4%
Total	100%	100%

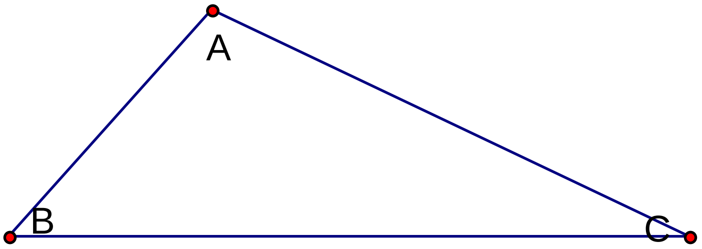
4. Cut out the two identical copies of the triangle below.

a. Use the two triangles to make a parallelogram, matching the short sides. Trace it.

b. Make and trace another parallelogram by matching the long sides of the two triangles.

c. Find the area of each parallelogram in square centimeters.

Possibly useful formula: area of a parallelogram = base x height



In this analysis, “Correct” means that the student computed the areas of both parallelograms and indicated that the two areas must be the equal. An “Approximately correct” answer either found two areas with small differences, but did not discuss the discrepancies, or only constructed one parallelogram and correctly found its area. Less than 10% of the “before” students, and 40% of the “after” students could even find the area given the formula. Many of the others multiplied the side lengths to find two wildly different areas, not noticing that the two areas must be equal. We believe that this does not show a lack of conservation of area, but, rather, a focus on formulas and procedures to the exclusion of conceptual understanding.

Two Triangles

	before	after
Correct	2%	8%
Approximately correct	7%	32%
Incorrect	49%	44%
Not done	42%	16%
Total	100%	100%

5. A garden is a rectangle, 5 yards by 2 yards. If 4 cubic yards of topsoil is spread evenly on the garden, how deep will it be?

The garden problem was one of the most difficult on the test, requiring not only an understanding of length, area, volume, but also a small amount of algebra. Many of the incorrect responses did not get very far, some only showing a 2 by 5 rectangle. Some of those who persisted often guessed the operation, choosing 10ÅÄ4, 10×4, or $10 - 4$. The fact that the vast majority did poorly on this problem reflects unfavorably on the algebra instruction the students have had (at least three years in middle and high school), as well as their background in measurement.

Garden Depth

	before	after
Correct	7%	16%
Incorrect	89%	76%
Not done	4%	8%
Total	100%	100%
Depth in cubic units	15%	24%

The last question involves ratios and volume.

6. Which chocolate milk recipe will taste more chocolaty? Why?

Recipe	Milk (cups)	Chocolate powder (spoons)
A	3	8
B	2	5

The recipe problem is quite tricky. The units for milk and chocolate are different, and it is not clear how many spoons make a cup. Furthermore, when making chocolate milk, there is much more milk than chocolate, but the numbers in the recipe are smaller for milk because the units are bigger. Incorrect methods students used included considering only the amounts of milk, or amounts of chocolate, or both (without using ratios), attempting to find the percentage of chocolate in the mixture, and performing division with remainder. Some students computed correct ratios, but chose the wrong conclusion: for example, claiming a larger ratio of cups of milk to spoons of chocolate means a more chocolaty mixture.

Chocolate Milk

	before	after
Correct	26%	40%
Incorrect	72%	60%
Not done	0%	0%
Total	98%	100%

Discussion

The instrument shows the extremely weak understanding of measurement in the “before” students, and the still weak performance of “after” students in the more difficult problems, those that require more than one step or the coordination of several concepts.

All of the students had passed several years of algebra classes. While the instructors, state officials, and textbook writers usually claim that their instruction teaches concepts, these results suggest that most students have learned only isolated, ritualized procedures.

We teacher educators tend to assume that our students have experience with physical measurement, and understand the concepts of length, area, volume, weight, speed, etc. This may be true in some countries, but in the U.S., students often have only a fuzzy conception of these ideas. To many of these students, area, volume, perimeter, and so on are only algebraic formulas. Some units of measure are familiar (usually the U.S customary system of inches, feet miles, pounds, etc.), but the metric system is, again, some list of facts to be memorized. Students often cannot reliably estimate quantities, such as how far they drove to school, the distance to the next building, or the length of a pencil in centimeters. Conversions between units are more facts and procedures to memorize.

Other than an increasing emphasis on formalism, our curriculum has not changed radically in the last century. Daily life has. A hundred years ago, an average person needing clothing or furniture might make their own, or go to a tradesperson in their town. Now simply buying the materials to make a garment or a piece of furniture costs more than buying a readymade one from a discount store, imported from a developing country. Even cooking, a good way to learn practical measurement, is increasingly rare. The movie “Little Miss Sunshine” portrays a typical (?) family dinner in the U.S.: a big bucket of takeout fried chicken and bottles of soda: no cooking or measuring was necessary. Even when people do measure things, it is often with a digital device. Modern children think of a clock as a digital readout.

Two of the authors of this paper (Addington and Dennis (2007)) are writing a textbook for preservice elementary teachers based on measurement for a foundation of number, inspired by the Russian elementary curriculum Davydov et al. (1999) and its American adaptation *Measure Up* (Doughtery, et al. (2003)). The first chapter does not allow the use of numbers, forcing students to confront the basic concepts of measurement rather than relying on formulas. Later, students are led to deconstruct their understandings of mathematics by activities such as “Use only rice and measuring cups in sizes of 1, 1/2, 1/3, and 1/4 cup to divide $1\frac{1}{2}$ by $\frac{1}{4}$.” Preliminary experiences in class show that some topics need just a few experiences, while others require repeated exposure in different contexts. Mikens (2007) showed that high school geometry students' abilities in measuring with a ruler improved markedly after a few short lessons, but understanding concepts of area and perimeter required much more time.

Conclusion

We have argued that measurement practice is critical for measurement concepts, and measurement concepts are essential for understanding most mathematics. Because students in industrialized countries rarely have the opportunity to learn and practice measurement outside of school, measurement must be taught in school, especially in the context of science and the arts. Mathematics teaching must also include measurement practice and concepts, and must explicitly make connections to multiplicative thinking.

Without measurement and multiplicative thinking, skills in algebra and higher mathematics are an architrave floating in midair, to borrow Wittgenstein's phrase. The edifice of algebra has not feet of clay, but no feet whatsoever. Without physical foundations and applications, algebra becomes simply a filter to sort out the obedient rule-followers from the symbolically disadvantaged, the rebellious, and the creative.

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