

“Concept Study”: Open vs. Closed Understandings of Mathematical Ideas

Brent Davis, *University of British Columbia*

I propose “concept study” as a potentially useful emphasis in pre-service and in-service mathematics teacher education. Derived from a synthesis of “concept analysis” and “lesson study,” concept study involves collective interpretation of mathematical ideas with emphases on examination and elaboration of implicit associations that infuse them.

In an attempt to hint at the conceptual complexity of some of the ‘simplest’ or ‘most basic’ of mathematical ideas, I recently posed the question “What is multiplication?” during a lecture to some 200 pre-service elementary school teachers. Two descriptors were called out in response: *repeated addition* and *grouping*. These were the answers that I had expected. I was so confident in this expectation, in fact, that I’d typed them into the next slide of the powerpoint presentation when preparing it the day earlier.

I had good reason to be confident. It’s proven to be remarkably stable across a variety of populations and situations. Whether dealing with pre-service or practicing teachers, or whether working at elementary or secondary levels, in the few dozen times I’ve posed it to groups of educators, the responses have always come back pretty much the same.

The issue, of course, is not that these interpretations of multiplication are wrong, it’s that they quickly become inadequate. They might be useful for interpreting actions on physical objects and whole numbers, but their utility starts to fail when one is faced with multiplying fractions, decimals, integers, imaginaries, vectors, matrices, polynomials ... the list goes on. As more contexts, number systems, and applications are considered, the idea that multiplication is a matter of grouping and repeated addition grow ever more inadequate. What, for example, might $^{-}5$ groups of $^{-}8$ look like? Or how does one add $(3 - i)$ to itself $(^{-}\sqrt{2} + 4i)$ times?

To be honest, I was somewhat taken aback when I first began to study the limited and limiting responses that teachers provided to questions of this sort. However, I soon noticed that, in the contexts of doing mathematics with one another and with their students, most of these same teachers seemed to enact much richer, more nuanced understandings of the concepts at hand. I have also observed that inviting teachers to be more explicit about their enacted knowledge can have powerful effects on their understandings, beliefs about mathematics, confidence with the subject matter, and teaching practices.

In this paper, I outline what I have learned so far about these issues. Throughout the

discussion, I am guided by Chevallard's (1991) observation that most bodies of knowledge aren't designed to be taught, but to be used. Consequently, in most domains, many of the elements that are vital to effective *knowing* (i.e., effective utilization of existing knowledge) are tacit. That is, they are simultaneously *known* (in the sense that they are enacted) and *unknown* (in the sense that they may not be available for conscious representation or interrogation)—as illustrated by the contrast between teachers' explicit definitions of multiplication and the more nuanced interpretations that are encountered in their classrooms.

The structure of this writing is as follows: I begin with the identification of three key distinctions, namely *mathematical* versus *mathematics*, *explicit assertion* versus *implicit association*, and *open definition* versus *closed definition*. I use these distinctions both to locate my research foci in the context of contemporary research into mathematics for teaching and to frame a discussion of my current studies of the topic—which, in the second section of this writing, I develop in this writing through the notion of *concept study*. I end with some speculations on where this sort of research might be headed.

1. Some Important Distinctions

1.1. Mathematical versus Mathematics

One of the most prominent distinctions in the professional literature for mathematics teachers is the 'process *versus* product' dyad, with *process* referring to actions and dispositions associated with *establishing* knowledge and *product* to *established* knowledge. On this count, mathematics courses have for centuries been focused on established product—and, more specifically, on ensuring that learners are able to demonstrate proficiency with a subset of explicit conclusions.

The distinction between *mathematical* and *mathematics* echoes the process/product dyad in many ways, but it is intended to soften the implied action/object or dynamic/fixed dichotomies that are often associated with process/product. Both *mathematical* and *mathematics* engender senses of action *and* object, transformation *and* stability. More specifically, in work with teachers, I use the word *mathematical* to refer to the disciplined mode of inquiry that involves, but is not limited to, noticing, characterizing, and extending cases of sameness, pattern, and regularity—a mode that is oriented toward abstractions and generalizations that hold in a variety of experiences and applications. Its complement, *mathematics*, refers to those abstractions and generalizations. In brief, the term encompasses the assumptions, definitions, propositions, and conclusions that arise from a mathematical attitude, both explicit and implicit.

1.2. *Explicit Assertions versus Implicit Associations*

Working with teachers, mathematicians, and other mathematics education researchers, we have subdivided *mathematical* and *mathematics* into other categories that we've found useful for discussing what teachers need to know about the discipline in order to teach well. I provide a summary of some of the further distinctions in Figure 1, emphasizing that these subcategories are not intended as an accurate depiction of the 'way things are.' Nor do I mean to suggest that the categories presented are exhaustive. They are, rather, useful ways of talking about aspects of what is expected to happen in a contemporary mathematics classroom.

For example, drawing a distinction between *global strategies* and *local tactics* in discussions of *mathematical* has proven to be quite productive. In the context of ongoing work with teachers, *global strategies* is used to refer to those qualities that reach across contexts and levels of mathematical inquiry. Examples include the oft-cited abstraction and generalization. Its complement, *local tactics*, refers to tools and methods that are more specific, suited to making sense of particular situations or solving particular problems. (Examples are included in Fig. 1.)

However, for my immediate purposes, two subcategories of mathematics—that is, *explicit assertions* and *implicit associations*—are of more relevance. As presented in the right-hand columns of Figure 1, in the context of school mathematics, explicit assertions are loosely equivalent to what is popularly understood as “math.” They are those definitions, concepts, and competencies that are laid out in explicit terms in treatises, textbooks, tests, and curriculum documents. More cogently, the suggestion that multiplication is a matter of grouping and repeated addition is an example of an explicit assertion.

Explicit assertions represent just the tip of the iceberg of human knowing, as has been foregrounded by research into both processes of individual cognition (cf. Lakoff & Núñez, 2000; von Glasersfeld, 1995) and the history of collective knowledge (cf. Katz, 1986; Mazur, 2002)—a point that is of particular relevance in the next section when I develop the idea of *concept study*. The hidden part of the iceberg is manifest in associations that have been deliberately knitted into language by mathematicians and pedagogues, in accidental connections among idiosyncratic experiences, and so on. With regard to school mathematics, implicit associations comprise metaphors, images, analogies, gestures, examples, illustrations, and classroom exercises that are offered by teachers (and, often, students) as means to make sense of mathematics concepts.

Mathematical Doing (process; establishing knowledge)		Mathematics Knowing (product; established knowledge)	
A particular, disciplined mode of inquiry that involves noticing and extending cases of sameness, pattern, and regularity. It is oriented toward abstractions and generalizations that hold in a variety of contexts.		An evolving, but relatively stable collection of procedures, concepts, and conclusions that have emerged through centuries of mathematical inquiry. Different systems mathematics can (and do) exist.	
Global Strategies	Local Tactics	Explicit Assertions	Implicit Associations
Dispositions/aims/tendencies that are relevant to most contexts, including: Organizing; Abstracting; Generalizing; Justifying, Verifying & Refuting; Naming	Topic-specific tactics for making sense of situations, problems, etc., including: Try a simpler case; Ask what changes and what stays the same; Examine the converse	The sorts of ‘objects’ that are typically seen as “math,” including: Definitions; Properties; Techniques; Theorems; Symbols & Notations; Justifications; Exemplars	Connections (usually tacit) infusing concepts, including: Metaphors; Images; Analogies; Gestures; Examples; Illustrations; Exercises

Figure 1. Some categories of mathematics knowing

1.3. Open Definitions versus Closed Definitions

Of course, the problem with studying implicit associations is that their power is in invisibility. It’s difficult to notice them because they are such integral, enacted parts of knowing—and, conversely, attending to them places demands on consciousness that detract from their utility.

One method that has proven effective in efforts to produce more nuanced definitions of mathematical concepts is to pose more pragmatically phrased questions, such as “How does the concept of multiplication unfold through the K–12 curriculum?” (as opposed to “What is multiplication?”). As reported elsewhere (Davis & Simmt, 2006), this sort of rephrased question can work to deflect attentions away from explicit definitions and toward ranges of association that are invoked in school mathematics, especially when teachers of all grade levels participate in the discussions. In more recent work, however, I’ve come to realize that the strategy isn’t infallible. In a few situations I’ve run into a desire for clear and unambiguous answers. In the case of the lecture mentioned at the start of this writing, for example, the pre-service elementary teachers readily acknowledged the inadequacy of their preliminary response to the “What is multiplication?” question, but they were deeply dissatisfied with an elaborated response that was given in terms of other associations (e.g., hopping along a number line, generating an array, changing dimensions, stretching a number line). The complaint, voiced by a number of students, was that such a response left people at sea. As one phrased it, “You can’t just tell students what multiplication is *like*, you have to teach them what multiplication *is*. So ... what is it?”

It was this query that prompted a distinction between *closed definitions* (i.e., precise descriptions of a form or concept) and *open definitions* (i.e., context-specific characterizations

that are understood to evolve as new applications and associations arise). This proved to be a useful discernment to address the issue presented in the previous paragraph—and, in fact, it was subsequently applied to a range of topics, including the terms *mathematics* and *teaching*.

2. Concept Study

2.1. Mathematics-for-Teaching

Of course, space prohibits me from taking on questions such as “What is math?” and “What is teaching?”, but it does bear mention that these issues are far from settled despite the veneer of consensus that often permeates discussions of school mathematics. For the purposes of this writing, I will slide past these profoundly important topics with the mention that they should probably be understood in terms of context-dependent, constantly evolving, open definitions.

Mathematics-for-teaching, it follows, falls into the same category. Like many others, I have for the most part used a working definition along the lines of “the mathematical understandings that effective teachers (need to) know”—and, like many others, I agree that *mathematics-for-teaching* is more than a catalog of insights. It also entails, for example, an attitude of curiosity and disposition toward attending, with regard both to mathematics in general and to individual construals of collectively sanctioned mathematical truths. These points are underscored in the collaborative working sessions with practicing teachers, which are typically organized around topics that they have selected for inquiry. We always manage to focus on those topics, but over the course of our shared work there is always considerable ‘spillage’ into issues such as how math evolves, how people learn, why certain concepts are taught, and how school mathematics is entangled with issues of equity and justice.

Another topic is the role of teachers and children in shaping formal mathematics. With much of our attentions focused on how teachers and their students co-select the images, analogies, experiences, applications, and so on, part of the discussion inevitably moves toward how such selections help to frame cultural mathematical understandings and collective mathematical possibilities. In other words, teachers come to see themselves as participants in the production of mathematics—not so much because they are producing ‘new’ mathematics (in the sense that research mathematicians produce mathematics), but through the transformations to mathematical notions that occur through their efforts to render those notions learnable.

It is for this reason that I join the argument that *mathematics-for-teaching* is a legitimate branch of mathematical inquiry. It is a site of invention, not merely replication, one that is

conscious of not just mathematical knowledge, but insights from cognitive science (inc. neurology, linguistics, psychology, and other domains), history, cultural studies, and ecology.

2.2. *Concept Analysis + Lesson Study*

So how does one go about organizing a research program around the complementarities of mathematical/mathematics, explicit-assertions/implicit-associations, and closed-definition/open-definition, all embedded in contexts of working with teachers around matters of their disciplinary knowledge while welcoming a certain transdisciplinarity?

It occurred to us early on that we were actually doing a sort of *concept analysis*, akin to much of the work done through the 1970s, 80s, and 90s (see, e.g., Leinhardt, Putnam, & Hatrup, 1992), especially around the topic of multiplicative structures (Behr & Harel, 1990; Vergnaud, 1983). Other influences include error analysis (Radatz, 1979), with its emphasis on where and why learners' understandings fall short and, of course, studies of the figurative substrate of much of mathematics (e.g., Katz, 1986; Lakoff & Núñez, 2000).

In our work, these theoretical influences and foci of concept analysis are coupled to collaborative and interactive structures that are reminiscent of *lesson study* (e.g., Fernandez & Yoshida, 2004), as well as the explicitly recursive process through which ideas are articulated, developed, and tested: Set Goals → Analyze and Plan → Teach and Observe → Discuss and Revise → Teach and Observe → Discuss and Revise → Report on Goals → Set Goals, and so on. The shared work is, in the main, organized around collective examinations of selected concepts and curriculum emphases, seeking to make explicit the images, exemplars, analogies, and so on that they bring forward in their teaching and that are presented in curriculum materials. Once made explicit, teachers work together to develop open definitions of various topics, attentive to how ideas might be presented to their students in ways that help them be receptive to elaborations of concepts as they move through their grade school experiences.

2.3. *Progress to Date*

Progress is being made on both of the major aspects of the work—that is, with regard to both the structures for collaboration and the development of several open definitions on topics that include numbers systems, binary operations, and functions.

Regarding the collaborative work, a major emphasis has been on the deliberate incorporation of principles of complex co-activity (Davis & Simmt, 2003), focusing in particular on establishing sufficient redundancy among participants while honoring diverse areas of interest

and expertise. On these counts, among the critical elements noted to date are that it is *much* more productive to work with cross-grade groups of teachers than with teachers working at the same levels and that it appears that secondary teachers learn at least as much about implicit associations from their elementary school colleagues as the other way around. (This point has come as a bit of a surprise because, according to concept studies undertaken to date, the bulk of implicit associations that are introduced over the course of grade school mathematics are presented in the mid- to higher grades.)

As for the development of open definitions, given the limitations on space for this brief paper, it is perhaps most efficient to portray progress by describing a question developed for an interview protocol that is based on work with teachers. One that has proven to be particularly fecund is a query about how students make sense of multiplication (see Fig. 2; protocols are also under development for number systems, addition, subtraction, division, exponentiation, and functions). In brief, interviewees are asked how they might help learners to understand what is going on in different multiplicative situations. The intention is not to move through the entire list of questions, but to go as far as the interviewee is able to provide substantive responses and to linger on the item(s) that first present difficulties. At this point, the interaction moves to a more conversational mode as interviewer and interviewee grapple together with how/why difficulties in understanding might arise.

The multiplication protocol has proven particularly interesting because it typically becomes apparent that few interviewees have ready interpretations for situations that go beyond relatively small whole numbers. When presented with “ 12×25 ,” a huge majority offer no new interpretations, with comments such as “You could say this is 12 groups of 25, but that’s not really something that any of the kids are going to be able to imagine,” or “This is where I’d just teach them the math [i.e., the accepted algorithm].” In other words, on the surface it would seem that they are unaware of how new images or metaphors, such as multiplication-as-grid-making or multiplication-as-area-producing, provide more powerful, further-reaching interpretations than multiplication-as-repeated-addition.

However, the shift from formal questioning to more relaxed conversation usually reveals something quite different. As teachers are asked to draw a picture, or as the questioner suggests an unexpressed interpretation, very often the discussion moves to a much richer, much more nuanced analysis of the concept at hand. If and when that happens, the interview usually

continues to subsequent items. When it doesn't happen, we move to a different topic.

This interview/conversation strategy is rooted in the conviction that efforts to assess personal understanding necessarily contribute to transformations to that understanding. Our goal is not to identify where people go wrong, but to highlight where and when interpretive strategies fall short. At these moments, we also aim to better understand how people work together to invent interpretive possibilities for mathematical notions.

We're going to be looking at multiplication of different numbers and values in this section. We'd like you to talk about how you would help your students to make sense of what's going on in each case. There are lots of ways you might frame your responses. For example, you might talk about how you'd introduce the concept, or what you might say if a student was having trouble understanding, or the sorts of difficulties you've noticed among learners.		
2×3	16×10.9	$C = \pi d$
8×4	0.009×0.008	$y = 2x$
12×25	$\frac{1}{2} \times \frac{1}{4}$	$+4 \times -2$
(large number) \times (large number)	$\frac{1}{5} \times \frac{3}{8}$	$-5 \times +6$
4×0.2	$1 \frac{1}{3} \times 2 \frac{3}{4}$	-3×-7
0.3×0.3	Area = base \times height	$-5.1 \times +0.004$

Figure 2. An example of an interview protocol: "Number and Quantity Sense"

3. Closing Remarks

I re-emphasize that the research discussed about is, at best, focused on a narrow slice of mathematics-for-teaching. However, I would hasten to add that this slice—implicit associations that infuse/underpin/frame/enable/constrain mathematical notions—is one that calls for considerably more attention than it is receiving at the moment. Among the emerging issues is the fact that, perhaps more than any other domain of inquiry, we transform the object of our examination in the process of studying it. This is certainly the case in the context of concept studies. It has also proven to be the case in efforts to assess teachers' understandings that are based on the emerging results of those concept studies.

Further, this point applies not only to teachers' understandings of mathematical concepts, but to their understandings of the nature of mathematics. For the most part, participants have a ready appreciation of the recursively elaborative nature of personal understandings. But the realization that collective mathematical knowledge obeys a similar evolutionary dynamic is a revelation to many—one that has prompted some to consciously and deliberately reposition themselves in relation to the subject matter. As one teacher recently put it, "I'm suddenly seeing that what we're doing goes way beyond just 'passing on knowledge'; we're creating insights that weren't there before."

References

- Behr, M., & Harel, G. (1990). Understanding the multiplicative structure. In G. Booker, P. Cobb, & T.N. de Merdicutti (Eds.), *Proceedings of the PME XIV Conference, Volume III* (pp. 27–34). Mexico: Consejo Nacional de Ciencia y Tecnologia, Gobierno del Estado de Morelos.
- Chevellard, Y. (1991). *La transposition didactique*. Grenoble: La Pensée Sauvage.
- Davis, B., & Simmt, E. (2003). Understanding learning systems: Mathematics teaching and complexity science. *Journal for Research in Mathematics Education*, 34(2), 137–167.
- Davis, B., & Simmt, E. (2006). Mathematics-for-teaching: An ongoing investigation of the mathematics that teachers (need to) know. *Educational Studies in Mathematics*, 61(3), 293–319.
- Fernandez, C., & Yoshida, M. (2004). *Lesson study: A Japanese approach to improving mathematics teaching and learning*. Mahwah, NJ: Erlbaum.
- Katz, V.J. (1986). Using history in teaching mathematics. *For the Learning of Mathematics*, 6(3), 13–19.
- Lakoff, G., & Núñez, R. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York: Basic Books.
- Leinhardt, G., Putnam, R., & Hatrup, R.A. (Eds.) (1992). *Analysis of arithmetic for mathematics teaching*. Hillsdale, NJ: Erlbaum.
- Mazur, B. (2002). *Imagining numbers (particularly the square root of -15)*. New York: Douglas & McIntyre.
- Radatz, H. (1979). Error analysis in mathematics education. *Journal for Research in Mathematics Education*, 10(3), 163–172.
- Vergnaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 127–124). New York: Academic Press.
- von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. London: The Falmer Press.