

**PRE-SERVICE STUDENTS' PARTIAL UNDERSTANDINGS OF ELEMENTARY
MATHEMATICS CONCEPTS AND PROCEDURES: REDESIGNING INSTRUCTION**

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ABSTRACT

What type of curricular content is most likely to foster effective elementary mathematics teachers? While there is a wide spread consensus that prospective teachers must understand the mathematics they will teach, there is less research and agreement as to how to achieve this level of knowledge and, to some degree, what this knowledge looks like. In order to determine the mathematics that should be included in preparatory coursework we examined constructed responses to 15-word problems (N=558) over two years drawn from pre-service students' mandatory content exam written midway through their methods course. We took digital images of all items with errors and coded for patterns using qualitative software. Twenty-seven percent of the students had considerable difficulty with the exam items in each of the first two years of administration. In the third year, when the exam was moved to the beginning of the year, this number grew to 43% of students. Many of the errors revealed a limited understanding of some of the *big ideas* (Schifter & Fosnot, 1993) in mathematics such as place value or the distributive property. The results are not surprising given the nature of these students' classroom experiences. Rather than focus on re-teaching *common mathematical content knowledge* (Ball & Bass, 2003) we argue for the importance of reconstructing this partial or limited knowledge of mathematical big ideas, models and strategies. We make the case that these students need increased instruction of the underpinnings of *specialized knowledge for teaching* in order to be able to participate fully in the regular methods class curriculum.

INTRODUCTION

Over the last two decades there has been a growing interest in teachers' mathematical content knowledge and its effect on instruction (Ball, 1990; Ma, 1999). In part this interest was generated by the difficulty teachers experienced implementing reform-inspired curricula. There was building evidence that teachers who were committed to implementation of the basic tenets of reform were often held back by their weak content knowledge (EEPA, 1990). At the same time, the discussion about the negative impact of weak teacher content knowledge was, quite rightly, expanded to include an acknowledgement that mathematics instructional knowledge is much more than simply knowing mathematics. The mathematical knowledge necessary for teaching that Shulman (1987) delineated included not only *content knowledge* and *pedagogical knowledge* (general management of teaching) but also *pedagogical content knowledge*.

Building on this literature Ball and her colleagues have undertaken an ambitious project to define and then to measure, on a large-scale, the knowledge necessary for teaching mathematics (Hill, Schilling & Ball, 2004). They hypothesized that the mathematical knowledge teachers have, and how they *hold* it, could be correlated with student achievement in mathematics. They re-conceptualized Shulman's definition conceiving of the mathematical *knowledge necessary for teaching* as having two main components: *common content knowledge* and *specialized content knowledge for teaching* (Ball & Bass 2003). The common content knowledge comprises the knowledge that non-mathematicians outside of the field of teaching might have for example, how to multiply 25×35 efficiently, reliably and accurately while specialized content knowledge for teaching includes knowing why the algorithm works, alternative algorithms that would be useful for children, and typical errors that they might make when multiplying. If it proves that effective instruction depends upon a combination of teachers' common content knowledge and their specialized content for teaching knowledge, how can pre-

service courses most effectively engender this knowledge?

We have a limited understanding of the kind of studies prospective students should take in order to teach mathematics well, that is, what combination of ‘methods’ and ‘content’ optimally support the development of prospective teachers’ mathematical teaching knowledge (Wilson & Berne, 1999). There is, nonetheless, a strong argument that prospective teachers must know what they are teaching, as Borko et al. (1992) forcefully recommended, (as a result of watching novice teachers struggle to teach mathematics), “...prospective teachers must be given the opportunity in their university course work to strengthen their subject matter knowledge” (p.219). In order to fulfill this mandate we need a body of research on “...what teachers know well and what they know less well...for leveraging resources wisely toward the improvement of teachers’ opportunities to learn mathematics.” (Ball & Bass, 2003, p. 13). We address the following questions:

1. What are the areas of elementary school mathematics about which prospective teachers typically have misconceptions?
2. What is the nature of these misconceptions and, are there patterns when we look across a large number of students?
3. What are the implications for an appropriate curriculum for prospective teachers?

THEORETICAL FRAMEWORK

While earlier studies have sometimes examined errors in mathematics in a limited and mechanistic fashion, we believe that ‘errors’ can be more robustly examined as roadblocks in adults’ learning of the *big ideas*, *mathematical models*, and effective flexible *strategies* in mathematics. Schifter and Fosnot (1993) coined the term big idea to mean the “central, organizing ideas of mathematics—principles that define mathematical order...they are connected to the structures of mathematics” (p.35). For example, in the development of multiplication and division concepts the inverse relationship between the two operations is a big idea. Students who

have not yet constructed this will solve problems such as $6 \times ? = 42$ through more the more labour intensive methods of trial and error, adding up, or skip counting. The idea of the inverse relationship is big because it opens up new, often more elegant methods. They define mathematical models are the organizational tools such as a ratio table or number line that students can use to represent and organize their thinking in order to explore relationships and solve problems. Mathematical strategies are the range of calculation approaches students use, for example, repeated addition to solve a multiplication problem.

In this analysis, errors offer us a window into those specific areas in elementary mathematics that prove particularly demanding or crucial for learning. We also view errors as evidence, in some instances, of students' 'fragile' knowledge in that they may be able to solve problems in one context while struggling with the same concept in a different context. Viewed through this lens errors, or partial knowledge, give us a more complex picture of the mathematical knowledge of our prospective teachers and of the particularly fruitful areas for instruction.

METHOD

Study Context. We teach in a small faculty of education (800 students) in Canada. All Primary/Junior (K-6) students write a high stakes mathematics content exam with the cut score set at 75%. Students write the exam in two hours at the end of the first term of their pre-service year. No calculators are allowed but manipulatives are provided including: centi-cubes, ruler, fraction strips, rectangular prisms, centi-metre graph paper, tape and scissors. Students who fail in November have an opportunity to retake the exam the following March. While there is a long history of certification exams in the United States (for an overview see Hill, Sleep, Lewis & Ball, 2007) certification exams are rare in Canada. This exam differs from many large-scale exams in the United States in that it is not multiple choice and therefore offers more detailed data on prospective teachers' mathematical methods.

Instrument (Exam) and Sample. The instrument consists of 15 constructed-response word problems set at the Gr. 6-7 level (eleven and twelve year-olds) in accordance with the provincial curriculum. The problems are designed to assess prospective teachers' understanding of the big ideas in mathematics at this level as well as typical misunderstandings of mathematical procedures or algorithms. In 2004 and 2005 a total of 558 students wrote the exam.

Exam Preparation Procedures (2003-2005). The commercial problem solving *Deck D* (Seymour, 1980) and a mock exam were available for two months prior to the exam. In addition, approximately 45 students paid a fee to take the 16-hour tutoring course covering the exam topics each year. Some students also took private tutoring or joined study groups.

Scoring & Analysis. All answers with errors were digitally photographed (1686 images) and imported into ATLAS.ti qualitative software. Each item was coded for the type of error made. The codes were created from the research literature on the particular concept as well as from repeated patterns that we had not expected.

RESULTS, ANALYSIS and DISCUSSION

Once coded we grouped all of the items under broad categories of error: *limited understanding, procedural error, concrete modelling or less efficient methods, and small calculation or factual errors* (see Table 1).

Table 1: Rates of Categories of Errors across Problem Items (N=558)

	Quotative	Division	Quotative	Division	Partitive	Division	Rate	Whole	Unknown	Division of	Fractions	Fractions as	Operator	Addition of	Fractions	Ratio	Percentage	Area	Volume	Probability	Probability	Mean
Limited Understanding	5	4	4	4	4	4	22	15	4	3	5	3	11	11	4	3	7					
Procedures	9	6	1	17	9	9	10	1	16	35	1											1
Concrete		1	2	6							3											3
Modelling																						
Social Knowledge	13	4	5	14	4	8	1	9	3	1												3
Total Error	27	15	13	40	25	24	31	17	6	12	30	47	4	3	16							

Category 1: Limited Understanding. While there was a variety of limited understanding error rates across problems (3% – 22%), most fell into the five to seven percent range (see Table 1). Students who did not attempt a problem, made a brief but mostly incomplete attempt, attempted to use a procedure or algorithm but did not ‘set it up’ or model it properly, were included in this first category. For example, the student in Figure 1 is unsure how to ‘set up’ the division algorithm when posed with the problem: *The average car is 6 metres long. How many whole cars can you park in a single lane, bumper-to-bumper, in 1.4 km?* They multiply the size of the car 6m by 1000 (presumably ‘converting it to metric’) and then divide the result by the kilometers of the total parking space (1.4km) to find the number of cars that can be parked. Similarly, some students were unsure of the multiplier (see Figure 2) when posed with the problem: *A nickel is 0.2 cm thick. Paul has a stack of nickels worth \$5.25. How tall is the stack of nickels?* This student multiplied the worth of the coin stack by the thickness of the coins to find the height of the stack using an incorrect multiplier or ‘setting it up’ improperly. In these cases we speculate that the student would be unable to model the situation and translate that model into the appropriate use of the algorithm. During informal interviews when asked to model the problem with a diagram they report not knowing where to begin or alternatively draw an incomplete or incorrect diagram. They are therefore unsure of ‘what to divide by what’ as in Figure 1.

Figure 1: Unsure of multiplier or divisor (PD 197)ⁱ

9. Car = 6m
lane = 1.4Km

1000m = 1Km
move (-) 428.57

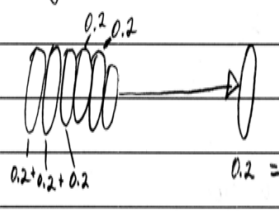
∴ you can fit 428 cars in 1.4Km, bumper to bumper

$$\begin{array}{r}
 14 \overline{) 60000.0} \\
 \underline{56} \phantom{x \phantom{x }} \\
 340 \\
 \underline{28} \\
 120 \\
 \underline{112} \\
 80 \\
 \underline{70} \\
 100 \\
 \underline{98} \\
 2
 \end{array}$$

PD 169

Figure 2: Unsure of the multiplier or divisor (PD 169)

(Thick) = 0.2 cm
(value) = 5.25
(height) = ?



∴ Multiply: 5.25

$$\begin{array}{r}
 \times 0.2 \\
 \underline{1050} \\
 0000 \\
 \hline
 10.50
 \end{array}$$

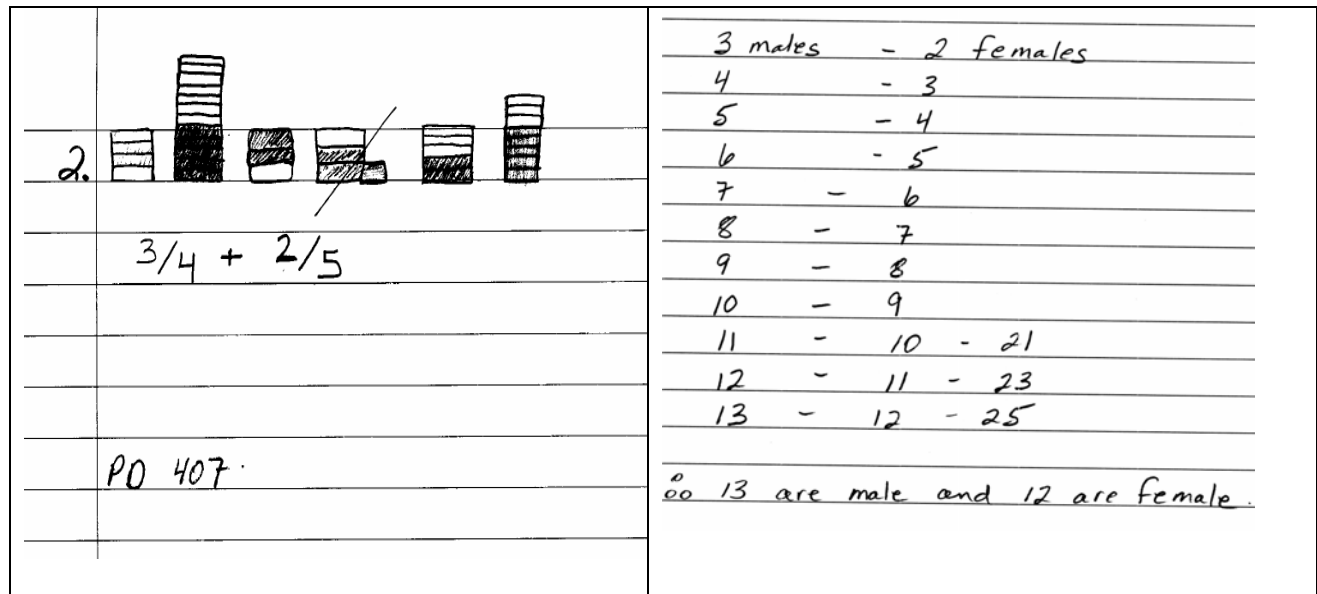
∴ The height is 10.5 cm tall

Finally, solutions which reflect a fundamental misunderstanding of the mathematical big ideas underpinning the concepts involved were included in this category. For example to solve the problem: *Of the following fractions: $\frac{3}{4}$, $\frac{5}{12}$, $\frac{2}{3}$, $\frac{3}{2}$, $\frac{2}{5}$, $\frac{5}{8}$, which two fractions (and only two) will give a sum that is less than 1?* students struggled with the big idea that *the whole matters: fractions are relations* (Fosnot & Dolk, 2007) often comparing fractions with different wholes (see Figure 3). Or, as another example they used additive reasoning to solve a multiplicative situation in the ratio problem: *There are three female for every two male professor on faculty. If there are 20 professors in total how many are women and how many are men?* as in Figure 4 misunderstanding the big idea that *for equivalence the ratio must be kept constant* (Jacob & Fosnot, 2007).

ⁱ The items have been faithfully reproduced from the originals in order to maintain anonymity. Each item is identified by a Primary Document number matched to a specific exam and indicating its digital storage location on a hard drive. If two pictures have the same PD number they are from the same exam.

Figure 3: Relationship not Understood (PD 407)

Figure 4: Relationship not Understood: Additive Thinking (PD 558)



Category 2: Difficulties with Algorithms and Procedures. We classified errors under this category when students chose an appropriate algorithm or procedure, set it up properly, but were unable to execute it fully, or make sense of their own work. Some traditional algorithms or procedures proved particularly difficult for students: division with remainders, decimal operations, multiplication and division of fractions, surface area and volume. There was a large variation in the misinterpretation of the steps or the meaning of the algorithm or procedure. On one end of the spectrum some students did not make sense of their answers. For example, in answer to the number of possible cars that could be parked in 1.4km a student calculated 233R2 cars (Fig. 5). It was nonsensical answers such as this that were part of the impetus for the original reform movement. We continue to find evidence of mathematics as an entirely rote and nonsensical endeavour in some of our students' answers.

On the other end along the spectrum errors reflect a misunderstanding of the 'rules' or steps but an attempt at sense-making. For example, in Figure 6 where the student reports that it would take her 7.13 days or 7 days 13 hours and 12.8 (remainder) minutes to give away \$11,000 at a rate of one dollar a minute. While this is incorrect, the student is trying to make some sense

of the quotient in the context of the question.

Figure 5: Procedural remainder error (PD221)

$$1400\text{m} = 1.4\text{ Km} \quad 6 \overline{)1400} \begin{array}{r} 233 \text{ R}2 \\ \underline{12} \\ 20 \\ \underline{18} \\ 20 \\ \underline{18} \\ 2 \end{array}$$

You can park 233 R2 cars in 1.4 Km

Figure 6: Remainder Error (PD114)

$$\frac{11,000}{\pi} = \begin{array}{l} 1 \text{ dollar per minute} \\ 60 \text{ dollars an hour} \\ 34 \\ 60 \\ 1440.00 \\ \# 1440 \text{ a day} \end{array}$$

$$1440 \overline{)11000.000} \begin{array}{r} 713 \\ \underline{10800} \\ 2000 \\ \underline{1440} \\ 5600 \\ \underline{4320} \\ 12800 \\ \underline{11520} \\ 1280 \text{ R} \end{array}$$

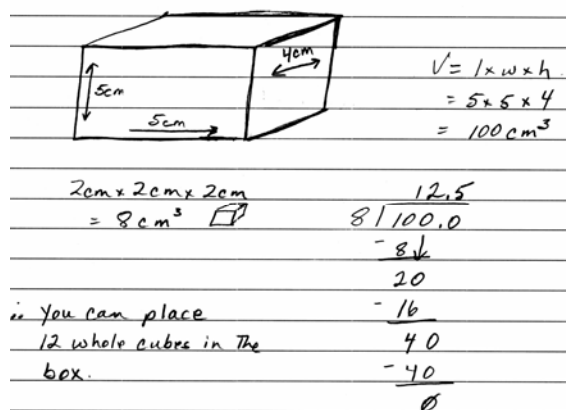
\therefore it would take 7 days 13 hours and 12.8 minutes.

It is not surprising that some of our students misunderstood the meaning or purpose of particular steps or answers in various algorithms. As many researchers have stated, algorithms are efficient because they simplify and mask complex procedures and mathematics (Ma, 2004). More disturbing than misunderstanding or incorrectly executing the algorithms, we believe that students' usage of these procedures has also interfered with the development of their mathematical knowledge of other big ideas. For example, students had difficulty using the decimal rules to solve problems such as: *A nickel is 0.2 cm thick. Paul has a stack of nickels worth \$5.25. How tall is the stack of nickels?* They divided 5.35 by 0.05 using the procedure of 'moving the decimal to the right to get rid of it in the divisor' often misplacing the decimal. We think that this method was a roadblock to their understanding that multiplication of the divisor by 100 shifts, not the decimal, but the five from five 100ths, to five 10ths, to 5 ones. As a result they have not constructed the big idea that *place value patterns occur when multiplying by 10* (Fosnot, 2007). In addition to blocking the development of big ideas in mathematics the rote learning of algorithms may also have interfered with the development of more flexible, sometimes simpler, methods for solving problems. Using the same example of dividing \$5.25 by .05 to find the number of nickels, students who had difficulty with the division of decimals did not think about

how many nickels in a dollar and therefore, how many in \$5.25? When students used this strategy the dispense with division by decimals. When we have discussed this method in interviews most students are able mentally to calculate the answer, it is not a matter of inability rather, the rigid adherence to the standard algorithm seems to interfere with access or the development of other methods. Certainly many researchers have made the case that the way that algorithms are traditionally taught interferes with children's mathematical learning and capacity (see for example Anghileri, 2001; Kamii & Dominick, 1998). While 'buggy' or misapplied algorithms might be viewed as a constructive stage through which children will pass (Smith, diSessa & Roschelle 1993) these adults have not moved beyond this stage -- at least in these instances; moreover, we believe they have been stifled in their development of foundational big ideas and other more flexible strategies.

There was a limited evidence of models in the first two categories of errors and, what models do appear, seem to have been learned in a rote fashion. The models of the situation for the measurement problems for example have not become 'models for thinking' (Gravemeijer & Stephan, 2002). Instead they are a rote procedure for many students. We hypothesize that students have been given these models -- as they are given algorithms -- rather than developing them out of a meaningful context. See for example the answer in Figure 7 to the problem: *You*

Figure 7: Spatial error with correct formula and diagram (PD418



have a box that is 4 cm wide by 5 cm deep by 5 cm high. You want to place cubes that are 2 cm wide by 2 cm long by 2 cm high in the box. What is the largest number of whole cubes that you could place in the box? Although the student has drawn the figures they have not used

them to check whether indeed 12 cubes could fit in the box. Most students drew the two

diagrams separately as we see here. When students drew the box with the cubes inside of it then they typically solved this correctly.

Category 3: Concrete Modelling and Less Efficient Methods. This category included solutions by students who used trial and error or made less use of traditional algorithms and instead attempted to model concretely the situation or to build or 'chunk' up to a solution. Many of these adults had participated in the tutoring course and had begun to try to work with mathematical ideas in a sense-making, if less efficient fashion. Students who were unsuccessful typically (although not always) showed a partial understanding of the mathematics and had not fully solved the problem because of poor physical drawing, correct modeling but with inefficiencies that increased the likelihood of error, or lack of proficiency with basic calculations.

Category 4: Small Calculation or Factual Errors. Students who made small calculation or 'social knowledge' errors (number of hours in a day or metres in a kilometer) were included in the last category. We were not concerned with these errors with the exception of the mislabeling of measurement problems that is, students who made labeling errors of cm or cm³ rather than cm² in the surface area problem or vice versa in the volume problem. Initially we viewed this as a small error linked to the testing situation like others in this category. However, we realized that this may also be an indication that the formula (for area or volume) is rote and meaningless to the student. Baturo and Nason (1993) found this to be the case in their interviews with pre-service students on eight area tasks. They found that when they interviewed students on measurement concepts that the 'dropping' the unit of measure cm or labeling it as volume (cm³) was linked to students' "rule-dominated understanding that had not been connected to concrete experiences and thus lacked any meaning" (p. 359). Certainly most students in tutoring and in interviews reported that they were unsure of what the numbers meant, or the terms *squared* and *cubed*.

Implications and Conclusion

We examined the results of 558 exams of 15 word problems set at the Gr. 6-7 level written by pre-service students midway through their one-year Bachelor of Education programme over two years (2004, 2005). Ninety of the 558 students had taken tutoring prior to the exam and all students were halfway through their 36-hour math methods course. About one quarter of the student body had difficulties with the exam (12% failures, 15% marginal passes). The following year (2006) the exam was taken upon entry into the programme. The rate of difficulty rose to 43% (21% failures and 22% marginal passes). We considered 75% - 84% as a marginal pass because examining profiles of students in this range we still find errors that indicate potentially larger problems. The limited understanding and procedures errors exemplified in figures: 2, 3, 5 and 6 were made by students in the marginal pass group.

The percentage of students with difficulties and the severity of some of the errors are not surprising as prospective teachers are themselves products of mathematical instruction, which has been implicated with poor results, at least in the United States (Battista, 1999). In Canada there are somewhat better results, for example, in large-scale assessments such as TIMSS (Robitaille, Taylor & Orpwood, 1996) when many of these students would have been in middle school. Nonetheless, there is some evidence that the criticisms of mathematics instruction as a rule-based endeavour focused on speed and memorization would have applied to many of these students' experience of school mathematics (O'Shea, 2003).

Given these results, what pre-service structure would best deepen this group's understanding of mathematics to a degree that they will be capable of undertaking the main tenets of the reform? Tenets outlined in the recent documents such as the National Council of Teachers of Mathematics Standards (NCTM 2000) or the recommendations in Ontario in the *Expert Panel Report on Teaching and Learning Mathematics Gr. 4-6* (Ontario Ministry of Education, 2004). To answer this question we return full circle to Ball, Hill and Bass' (2005)

theoretical description of what it means to know mathematics for teaching cited in the introduction. They described an amalgam of mathematical content knowledge that is both common content knowledge and specialized knowledge for teaching. We use their example found in the *American Educator* (Ball, Hill & Bass, 2005, pp. 17-18) to explore our thinking. In this example, Ball et al. examine double-digit multiplication delineating the common knowledge aspects and the specialized knowledge for teaching teachers must have to teach this concept well. We briefly summarize their longer discussion in Table 7.

TABLE 2: Double Digit Multiplication Example of Teachers' Mathematics Content Knowledge

Common Content Knowledge	Specialized Content Knowledge of Mathematics for Teaching
-reliably use the multi-digit algorithm	<ul style="list-style-type: none"> - understand the meaning behind the steps in the algorithm e.g. why one 'slides' the second partial product over one column - use appropriate representations such as an area model of $(20 + 5) \times (30 + 5)$ - diagnosing typical student errors - connecting the traditional algorithm to the area model - choosing appropriate numbers or examples (in this instance one that may or may not involve regrouping).

Initially, as we examined the results we felt that our instruction needed an increased focus on developing the students' common content knowledge of mathematics. However, a close examination of the types of errors students made leads us to believe that a sizeable group of students do not have a sufficient knowledge base to grapple with a methods course focused on learning the types of knowledge described under specialized knowledge for teaching. Further, that a focus on strengthening common content knowledge will not offer a sufficient foundation to do so. Instead, we suggest students need further instruction in the fundamental knowledge *underpinning* these packets of specialized knowledge for teaching. For example, a sizeable segment of our student body could not participate fully in a discussion and examination of *the*

meaning behind the steps in the algorithm (Table 7) without previously or concomitantly constructing a number of big ideas. First, students must understand single-digit multiplication, that is, be able to think about the number of groups and the number in the group simultaneously (Clark & Kamii, 1996) or the big idea of ‘unitizing’ (Fosnot & Dolk, 2001). Multiplicative reasoning is not just repeated addition as many students were taught. Second, students must have constructed an understanding of the distributive property of multiplication (Ma, 1999) that is, that 25×35 can be thought of as $25(30 + 5)$ or $(20 + 5)(30 + 5)$ laying the groundwork for partial products. Finally students must understand the big idea of place value including recognizing the patterns that occur when multiplying by 10. Continuing with the example, in order to *use appropriate representations such as an area model*, students must understand the often illusive relationship between linear units and area units (Kamii & Kysh, 2006).

We know that a sizeable group of our students (43%) has difficulty with some of these fundamental big ideas of: unitizing, the distributive property of multiplication, place value patterns and the relationship of linear units to area units. An increased focus on pre-service students’ ability to use the algorithm reliably will not give them the foundational big ideas, strategies and mathematical models they need in order to develop specialized knowledge for teaching multi-digit multiplication. While most of these students either have, or will eventually, pass our content exam and be certified to teach, they will do so without having fully participated in the discussion of pedagogy that presently dominates our instruction. We are therefore shifting our curriculum to include significant opportunities and expectation that students reconstruct and deepen their knowledge of mathematics from early concrete models, through intermediate stages, eventually to efficient algorithms -- similar to recommendations of instruction for elementary and intermediate students found in work coming out of the Realistic Mathematics Education movement in the Netherlands (e.g. Gravemeijer, 2004; Gravemeijer & Stephan, 2002; Van Den Heuvel-Panhuizen 2003) and taken up in the United States by Fosnot & Dolk (2001).

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