INQUIRY INTO STUDENTS’ MATHEMATICAL THINKING THROUGH ERROR ANALYSIS AS A MEANS TO TEACHER DEVELOPMENT

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Abstract Recent research suggests that the idea of teachers understanding students’ thinking has been widely promoted and supported in the education community. However, the link between error analysis which is an important way to inquire into students’ thinking and mathematics teachers’ professional development is largely unexamined, particularly at the secondary level. In this paper, using the Mathematical Error Analysis Questionnaire constructed by the author and corresponding interview questions, based on the error analysis framework, it presents the results of four levels engaging in students’ thinking for teachers and the general description of error analysis as a professional skill of the investigated teachers. The results of the analysis suggest that the findings of four levels help us to better understand mathematics teachers’ professional skills so as to enhance teacher development, while the dual nature of mathematical error analysis is different sides of the same coin.

Key Words error analysis; mathematical error; middle school mathematics teacher; professional development

Introduction

In the context of world wide educational reforms that require teachers to understand and respond to student thinking. In the Professional Standards (National Council of Teachers of Mathematics [NCTM], 1991), the analysis of students’ thinking is highlighted as one of the central tasks of mathematics teaching. Recent research suggests the analysis of students’ thinking is seen as a resource that can help teachers make informed decisions in their classrooms and improve their practice (Doerr, 2006; Kazemi & Franke, 2004; Steinberg, Empson, & Carpenter, 2004), so inquire into students’ thinking has been seen a means to teacher development. Although there are various ways in which teachers can understand their students’ mathematical ideas, the link between error analysis which is an important way to inquire into students’ thinking and mathematics teachers’ professional development is largely unexamined, particularly at the secondary level, this study attempts to contribute to this domain.

This study attempts to investigate Chinese teachers’ professional skill in mathematics and pedagogy within a cultural context through error analysis, to explore how this skill is used by teachers to understand and develop students’ mathematical thinking. A group of middle-school teachers in China will be included in this study, using the Mathematical Error Analysis Questionnaire constructed by the author and corresponding interview questions. And the question that provides the focus for this study is: How do the middle school mathematics teachers engage in students’ mathematical thinking in error analysis?

Conceptual Framework
The framework that guided much of this study is based on Error Analysis Framework which focuses on the quality of thinking, the four phases of the process and the angle view of error analysis (Peng, 2007). This framework is depicted in Figure 1.

**Figure 1.** Error Analysis Framework

In the Error Analysis framework, the quality of thinking includes profundity, deepness, flexibility, doubt, quickness, originality, while identifying, explaining, evaluating, and correcting students’ mathematical errors of the four phases, and the angle of mathematical, logical, psychological, and cultural view.

In this study, the Error Analysis Framework guided data collection, analysis, and reporting.

**Methodology**

**Sample**
The study involved 25 middle school mathematics teachers in China from seventh to ninth grade at the middle school level. Criteria for inclusion of teacher volunteers in the study is: (1) current teaching of mathematics in seventh to ninth grades; (2) teaching in school districts that have typical characteristics with respect to the students’ ethnic, economic, and cultural diversity; (3) willing to provide the data relevant to the reliability and validity of this study, including interviews.

**Procedures**
Data were collected with an author-constructed Mathematical Error Analysis Questionnaire and interviews with selected teachers.

**Mathematical Error Analysis Questionnaire**
The questionnaire consisted of four problems that were designed to examine teachers’ skill in error analysis in algebraic and geometric topics of radical expression, equation, triangle and quadrangle. Each of the four problems focused on one aspect of teachers’ skills through identifying, addressing, diagnosing, and correcting students’ mathematical errors.

**Interviews**
The objective of the interviews was to examine teacher’ skills to further explore how this skill was used by teachers to understand and develop students’ mathematical thinking.

**Data Analysis**

**Questionnaire**
Qualitative analysis method was used in the analysis of the Mathematical Error Analysis Questionnaire, and 4 different categories consisted of identifying, addressing, diagnosing, and correcting students’ mathematical errors was identified which included the responses to the four problems. The responses were categorized into groups and assigned a descriptive code. Two researchers used the resulting codes to analyze the responses independently. Both sets of codes were compared, and then, through discussion with the third researcher, the disparities were reconciled to reach valuable agreements on the responses. The coding of the four problems of questionnaire is used by P1, P2, P3, and P4.

**Interviews**

Transcriptions were made of the interviews. The transcriptions were coded using the 4 response categories consisted of identifying, addressing, diagnosing and correcting students’ mathematical errors. The coding of the investigated teacher is used by six codes ABCDEF, which represents age, gender, degree, teaching years, school district and confidence about evaluating students’ mathematical thinking respectively.

**Findings**

**Four levels of engagement with students’ mathematical thinking in error analysis**

It was found that there are four different levels of engagement with students’ mathematical thinking in error analysis, which can be named by ‘goose egg’ analysis level, indistinct analysis level, partly exact analysis level and totally exact analysis level respectively (see table 1).

<table>
<thead>
<tr>
<th>Level 1</th>
<th>Can not at all or very lightly identify the mathematical errors of students.</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>Can not find the reasonable component of students’ mathematical errors; can not find the reasons for students’ mathematical errors or simply attribute to students’ mathematical errors to such as “non-seriousness of learning” and “poor basic knowledge and basic techniques.”</td>
</tr>
<tr>
<td></td>
<td>Can not evaluate the influence of students’ mathematical errors on their latter learning, can not evaluate students’ learning levels according to their mathematical errors.</td>
</tr>
<tr>
<td></td>
<td>Can not present correcting teaching strategies for students’ mathematical errors.</td>
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</table>

<table>
<thead>
<tr>
<th>Level 2</th>
<th>Can identify the mathematical errors of students but can not find the underlying reasons.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Can not find the reasonable component of students’ mathematical errors, can explain the reasons for students’ mathematical errors, but only consider them as they stand, can not analyze them for relative high level theoretical point of view.</td>
</tr>
<tr>
<td></td>
<td>Can simply evaluate the influence of students’ mathematical errors on their latter learning, can evaluate students’ learning levels according to their mathematical errors to a certain degree.</td>
</tr>
<tr>
<td></td>
<td>Can present correcting teaching strategies for students’ mathematical errors, but not so suit for the special cases.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Level 3</th>
<th>Can identify the mathematical errors of students and find the underlying reasons.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Can evaluate students’ mathematical errors from a reasonable point of view (a qualitative leap), can analyze students’ mathematical errors for a singly relative high level theoretical point of view.</td>
</tr>
<tr>
<td></td>
<td>Can rightly evaluate the influence of students’ mathematical errors on their latter learning, can evaluate students’ learning levels according to their mathematical errors in relatively right.</td>
</tr>
</tbody>
</table>
- Can present singly practical correcting teaching strategies for students’ mathematical errors, but not so suit for the special cases.

**Level 4**
- Can rightly and quickly identify the mathematical errors of students and the underlying reasons.
- Can understand the reasonability of students’ mathematical errors, can profoundly explain and analyze students’ mathematical errors form plenty of theoretical point of view.
- Can rightly evaluate the influence of students’ mathematical errors on their latter learning, can rightly evaluate students’ learning levels according to their mathematical errors.
- Can present many practical correcting teaching strategies for students’ mathematical errors.

To illustrate the four levels exemplified in the problems of the questionnaire, P1 and the interview transcription for P1 are introduced in the following.

(P1) In $\triangle ABC$, $AD$ is the bisector of $\angle A$ and $BD > DC$, prove $AB > AC$.

The following is a students’ proof (Pre-investigated data).

**Proof.** As is shown in Figure 2, by finding the point E on $AB$ such that $AE = AC$ and then connecting D and E, we derive $\triangle AED \cong \triangle ACD$, and then $\angle AED = \angle C$ from that $AD$ is the bisector of $\angle A$. From the external angle theorem, we deduce $\angle AED > \angle B$, and then $\angle C > \angle B$ which indicates $AB > AC$. Figure 2

According to the interview data, on this problem, there are four different levels of engaged with students’ mathematics thinking in error analysis for middle school mathematical teachers (See table II). The following are the two interview questions.

(1) Do you think the student’s proof is right? If not, point out the wrong(s) and analyze the possible reason(s) for this (these) wrong(s).

(2) Can you find some proof approach for this question? Please write them in brevity.

**Table II** Interview transcript for four levels of engagement with students’ mathematical thinking

**Level 1**
- 2fm06u2: it is correct.
  - $\triangle AED \cong \triangle ACD$ indicates $AE = AC$ and $AE < AB$ result in $AB > AC$.

- 4fb12u4: Basically correct. In the proof, however, the condition $BD > DC$ was neglected. Taking point D as the centre of the circle and CD as radius, draw arc which intersects AB or the extension of AB at point E, then the proof can be started from the condition $BD > DC$.

- 2ma03r4: it is wrong. The possible reason is that students always think that the angle towards a bigger side is bigger in a triangle.
  - Firstly, find point E such that $AE = AC$ for the bisector $AD$, then according to $AE = AC \Rightarrow \triangle AED \cong \triangle ACD \Rightarrow AE = AC$ and $AB = AE + BE$, infer $AB > AC$.

**Level 2**
- 3ma06r4: it is wrong. The reason is that the student doesn’t recognise the potential graph visually. The graph has many drawing ways.

- 2fb07r4: Not considered completely. there are two cases neglected, one is the superposition of B with E and the other is $AB = AC, BD = DC$ which contradicts with the condition $BD > AC$, consequently, B and E are not the same position.
Level 3  ● 2fb09r4: From the starting, the conclusion $AB > AC$ is potentially admitted. The reason is that the student doesn’t beyond the influence of geometry graphs on him. Moreover, he doesn’t find the reason for wrong when you tell him that the proof is wrong.

Level 4  ● 3fm12u4: Selecting point E on AB such that $AE = AC$ admit the wrong that to prove the conclusion by itself. This is typically a vicious circle which is a common wrong for students. This type of wrong often appears in students’ learning of geometrical argument, especially in the learning of the section of the properties and theorems of the bisector of an angle and the section of the definition and distinguishing of bowstring tangent angle.

General description of teachers’ skills in error analysis
Through the data analysis of questionnaire and interview, the general description of teachers’ skills in error analysis can be found (see Table III).

<table>
<thead>
<tr>
<th>Questionnaire</th>
<th>Void</th>
<th>Level 1</th>
<th>Level 2</th>
<th>Level 3</th>
<th>Level 4</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Num. %</td>
<td>Num. %</td>
<td>Num. %</td>
<td>Num. %</td>
<td>Num. %</td>
<td>Num. %</td>
</tr>
<tr>
<td>P1</td>
<td>0.0</td>
<td>6.0</td>
<td>24.0</td>
<td>36.0</td>
<td>8.0</td>
<td>32.0</td>
</tr>
<tr>
<td>P2</td>
<td>4.0</td>
<td>6.0</td>
<td>24.0</td>
<td>32.0</td>
<td>5.0</td>
<td>20.0</td>
</tr>
<tr>
<td>P3</td>
<td>1.0</td>
<td>4.0</td>
<td>4.0</td>
<td>44.0</td>
<td>8.0</td>
<td>32.0</td>
</tr>
<tr>
<td>P4</td>
<td>2.0</td>
<td>8.0</td>
<td>4.0</td>
<td>44.0</td>
<td>6.0</td>
<td>24.0</td>
</tr>
</tbody>
</table>

From the point of view of current reality status, as a whole, the levels of engagement with students’ mathematical thinking in error analysis for the mathematics teachers is relatively low; most of them are in the position of indistinct analysis level.

Discussion and Conclusion

On the significance of error analysis: implications for mathematics teacher education
The four levels of engagement with students’ mathematical thinking in error analysis was drawn from the research data, it not only described the present situation of middle school mathematics teachers’ competence in mathematics and pedagogy, but also reflected certain internal law’s characteristics of error analysis of mathematics teachers, it can help us to better understand mathematics teachers’ professional skills so as to enhance teacher development. What should be noticed are that the four levels are tied up and the later level is developed from the previous levels, and what we need to do in mathematics teacher education is to help teacher develop from the low level to the higher level.

On the dual nature of error analysis: different sides of the same coin
There is a dialectic viewpoint for error analysis, namely, right and wrong are not absolute two things but different sides of the same coin. If we understand the right why is right deeply, then we can understand deeply the wrong why is wrong. On the contrary, if we understand the wrong why is wrong deeply, then we can understand deeply the right why is right.
When we help mathematics teacher develop their professional skills in error analysis, we can provide mathematical task which is a new trend for teacher development for them (Zaslavsky, 2005; Zaslavsky, Chapman, & Leikin, 2002). When they understand mathematics deeply, their levels of engaged in students mathematical thinking will be high.

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References


