

# **THE KNOWLEDGE QUARTET: A THEORY OF MATHEMATICAL KNOWLEDGE IN TEACHING**

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*In this paper, a practice-based framework for the identification and discussion of prospective elementary school teachers' mathematics content knowledge is described. This framework - 'the Knowledge Quartet' - emerged from intensive scrutiny of 24 videotaped lessons, taught by novice teachers. Application of the Knowledge Quartet in lesson observation and teacher education is illustrated with reference to a particular lesson taught by one trainee teacher.*

## **INTRODUCTION AND RATIONALE**

It is now widely agreed that a vital component of the complex knowledge base for teaching is the transformation of subject matter knowledge (SMK) into a form that might enable others to learn it. The term 'pedagogical content knowledge' (PCK) was coined by Shulman (1986) to depict "the particular form of content knowledge that embodies the aspects of content most germane to its teachability" (p. 9). Questions about the adequacy of teachers' SMK have been voiced for more than a decade in the US (e.g. Kennedy, 1990) and the UK (e.g. Alexander, Rose & Woodhead, 1992). Specific concerns about elementary teachers' SMK and PCK have been a recurrent theme in reports by the UK government inspection agency, the Office for Standards in Education (e.g. Ofsted, 2003).

In the UK, most trainee teachers follow a one-year, postgraduate course leading to a Postgraduate Certificate in Education (PGCE) in a university education department. About half of the PGCE year is spent working in schools as an 'intern', under the guidance of a school-based mentor. Research shows that feedback on observation of lessons taught by student teachers typically focuses heavily on organisational features of the lesson, with very little attention to the subject-matter being taught (e.g. Strong & Baron, 2004). The purpose of the research reported in this paper was to develop an empirically-based conceptual framework for the discussion of the role of trainees' mathematics SMK and PCK, in the context of lessons taught on their school-based placements. Such a framework would need to capture a number of important ideas and factors about content knowledge within a small number of conceptual categories, with a set of easily-remembered labels for those categories. The research reported in this paper was undertaken in collaboration with colleagues Peter Huckstep, Anne Thwaites, Fay Turner and Jane Warwick. I shall use the pronoun 'we' in this text as a natural way of acknowledging their contribution.

Recent studies of Deborah Ball and her colleagues at the University of Michigan have been directed towards a “practice-based theory of knowledge for teaching” (Ball & Bass, 2003). The same description could be applied to our own study, but while parallels can be drawn between the methods and some of the outcomes, the two theories look very different. In particular, the theory that emerges from the Michigan studies unravels and clarifies the formerly somewhat elusive and theoretically-undeveloped notions of SMK and PCK. As a consequence, Shulman’s SMK is separated into ‘common content knowledge’ and ‘specialized content knowledge’, while his ‘pedagogical content knowledge’ is divided into ‘knowledge of content and students’ and ‘knowledge of content and teaching’ (Ball, Thames & Phelps, submitted). In our own theory, as will become apparent later, the distinction between different kinds of mathematical knowledge is of lesser significance than the classification of the situations in which mathematical knowledge surfaces in teaching. In this sense, the two theories may each have useful perspectives to offer to the other.

## **METHOD**

This study took place in the context of a one-year PGCE course, in which 149 trainees followed a route focusing either on the ‘lower primary’ years (LP, ages 3-8) or the ‘upper primary’ (UP, ages 7-11). Six trainees from each of these groups were chosen for observation during their final school placement. Two mathematics lessons taught by each of these trainees were observed and videotaped, i.e. 24 lessons in total. Following the lesson, the observer/researcher wrote a brief (400-500 words) descriptive synopsis of the lesson. From that point, we took a grounded approach to the data for the purpose of generating theory (Glaser & Strauss, 1967). In particular, we identified aspects of trainees’ actions in the classroom that seemed to be significant in the limited sense that it could be construed to be informed by their mathematics SMK or PCK. These were grounded in particular moments or episodes in the tapes. This inductive process generated a set of 18 codes. Next, we revisited each lesson in turn and, after intensive study of the tapes, elaborated each descriptive synopsis into an ‘analytical account’ of the lesson. In these accounts, significant moments and episodes were identified and coded, with appropriate justification and analysis concerning the role of the trainee’s content knowledge in the identified passages, with links to relevant literature.

Our catalogue of 18 codes presented us with the following problem. We intended to offer our findings to colleagues for their use, as a framework for reviewing trainees’ mathematics content knowledge from evidence gained from classroom observations of teaching. We anticipate, however, that 18 codes is too many to be useful for a one-off observation. Our resolution of this dilemma was

to group them into four broad, super-ordinate categories, or ‘units’, which we term ‘the Knowledge Quartet’<sup>1</sup>. A detailed methodological account is given in Rowland (in press).

## THE KNOWLEDGE QUARTET

We have named the four units of the Knowledge Quartet as follows: foundation; transformation; connection; contingency. Each unit is composed of a small number of cognate subcategories. For example, the third of these, *connection*, is a synthesis of four of the original 18 codes, namely: *making connections*; *decisions about sequencing*; *anticipation of complexity*, and *recognition of conceptual appropriateness*. Our scrutiny of the data suggests that the Quartet is comprehensive as a tool for thinking about the ways that subject knowledge comes into play in the classroom.

However, it will become apparent that many moments or episodes within a lesson can be understood in terms of two or more of the four units; for example, a **contingent** response to a pupil’s suggestion might helpfully **connect** with ideas considered earlier. Furthermore, it could be argued that the application of subject knowledge in the classroom *always* rests on **foundational** knowledge. Drawing on the extensive range of data from the 24 lessons, we offer here a brief conceptualisation of each unit of the Knowledge Quartet. A more detailed account is given in Rowland, Huckstep & Thwaites (2005).

The first category, *foundation*, consists of trainees’ knowledge, beliefs and understanding acquired ‘in the academy’, in preparation (intentionally or otherwise) for their role in the classroom. The key components of this theoretical background are: knowledge and understanding of mathematics *per se* and knowledge of significant tracts of the literature on the teaching and learning of mathematics, together with beliefs concerning the nature of mathematical knowledge, the purposes of mathematics education, and the conditions under which pupils will best learn mathematics. The second category, *transformation*, concerns knowledge-in-action as demonstrated both in planning to teach and in the act of teaching itself. As Shulman indicates, the presentation of ideas to learners entails their re-presentation (our hyphen) in the form of analogies, illustrations, examples, explanations and demonstrations (Shulman, 1986, p. 9). Of particular importance is the trainees’ choice and use of examples presented to pupils to assist their concept formation, language

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<sup>1</sup> Specifically, the codes contributing to each of the four units are as follows:

FOUNDATION: adheres to textbook; awareness of purpose; concentration on procedures; identifying errors; overt subject knowledge ; theoretical underpinning; use of terminology.

TRANSFORMATION: choice of examples; choice of representation; demonstration.

CONNECTION: anticipation of complexity; decisions about sequencing; recognition of conceptual appropriateness; making connections between concepts; making connections between procedures.

CONTINGENCY: deviation from agenda; responding to children’s ideas; use of opportunities.

acquisition and to demonstrate procedures. The third category, *connection*, binds together certain choices and decisions that are made for the more or less discrete parts of mathematical content. In her discussion of ‘profound understanding of fundamental mathematics’, Liping Ma cites Duckworth’s observation that intellectual ‘depth’ and ‘breadth’ “is a matter of making connections” (Ma, 1999, p. 121). Our conception of this coherence includes the sequencing of material for instruction, and an awareness of the relative cognitive demands of different topics and tasks. Our final category, *contingency*, is witnessed in classroom events that were not anticipated or planned for. In commonplace language it is the ability to ‘think on one’s feet’. In particular, the readiness to respond to children’s ideas and a consequent preparedness, when appropriate, to deviate from an agenda set out when the lesson was prepared.

### LAURA’S LESSON

We now proceed to show how this theoretical construct, the Knowledge Quartet, is being applied, by detailed reference to just one of the 24 videotaped lessons. The trainee in question, Laura, was teaching a Year 5 (pupil age 9-10) class about written multiplication methods. By focusing on just one lesson we aim to maximise the possibility of the reader’s achieving some familiarity with Laura and the children in her class, as well as the structure and the flow of her lesson.

The key focus of this lesson is on teaching column multiplication of whole numbers, specifically multiplying a two-digit number by a single digit number. After Laura has settled the class on the carpet in front of her, the lesson begins with a three-minute *mental and oral starter*<sup>2</sup>, in which the children rehearse recall of multiplication bonds. There follows a 15-minute *introduction* to the *main activity*. Laura reminds the class that they have recently been working on multiplication using the ‘grid’ method (see below). She speaks about the tens and units being “partitioned off”. Simon is invited to the whiteboard to demonstrate the method for 9x37. He writes:

$$\begin{array}{r|l|l} \times & 30 & 7 \\ \hline 9 & 270 & 63 \end{array} = 333$$

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<sup>2</sup> Conforming to National Numeracy Strategy guidance (DfEE, 1999), Laura segments the lesson into three distinctive and readily-identifiable phases: the *mental and oral starter*; the *main activity* (an *introduction* by the teacher, followed by *group work*, with tasks differentiated by pupil ability); and the concluding *plenary*.

Laura then says that they are going to learn another way. She proceeds to write the calculation for  $9 \times 37$  on the whiteboard in a conventional but elaborated column format, explaining as she goes along:

$$\begin{array}{r}
 37 \\
 \times 9 \\
 \hline
 270 \\
 7 \times 9 \\
 \hline
 333 \\
 1
 \end{array}$$

Laura performs the sum  $270+63$  by column addition *from the right*, ‘carrying’ the 1 (from  $7+6=13$ ) from the tens into the hundreds column. She writes the headings h, t, u above the three columns.

Next, Laura shows how to “set out”  $49 \times 8$  in the new format, followed by the first question ( $19 \times 4$ ) of the exercises to follow. The class proceeds to 24 minutes’ work on exercises that Laura has displayed on the wall. Laura moves from one child to another to see how they are getting on. She emphasises the importance of lining up the hundreds, tens and units columns carefully.

Eventually, she calls them together on the carpet for an eight-minute *plenary*. She asks one boy, Sean, to demonstrate the new method with the example  $27 \times 9$ . Sean gets into difficulty; he is corrected by other pupils and by Laura herself. As the lesson concludes, Laura tells the children that they should complete the set of exercises for homework.

We now select from Laura’s lesson a number of moments, episodes and issues to show how they might be perceived through the lens of ‘the Knowledge Quartet’. Some possibilities for discussion with the trainee, and for subsequent reflection, are flagged below thus: **Discussion point**. We emphasise that the process of *selection* in the commentary which follows has been extreme.

### Foundation

First, Laura’s professional knowledge – a case of what the Michigan team would surely call ‘specialized content knowledge’ - underpins her recognition that there is more than one possible written algorithm for whole number multiplication. We conceptualise this within the domain of fundamental knowledge, being the *foundation* that supports and significantly determines her intentions or actions. Laura’s learning objective seems to be taken from the National Numeracy Strategy (NNS) *Framework* (DfEE, 1999) teaching programme for Year 4:

Approximate first. Use *informal* pencil and paper methods to support, record or *explain* multiplication. *Develop* and *refine* written methods for TUxU (p. 3/18, emphasis added)

These objectives are clarified by examples later in the *Framework*; these contrast (A) *informal* written methods - the grid, as demonstrated by Simon - with (B) *standard* written methods - the column layout, as demonstrated by Laura in her introduction. It is perhaps not surprising that Laura

does not *question* the necessity to teach the standard column format to pupils who already have an effective, meaningful algorithm at their disposal. A best-selling handbook for trainee primary teachers, written by a respected teacher educator, explains why the standard algorithm works, but forcibly advocates the adequacy and pedagogical preference of grid-type methods with primary pupils (Haylock, 2001, pp. 91-94).

**Discussion point:** where does Laura stand on this debate, and how did this contribute to her approach in this lesson?

At this stage of her career in teaching, Laura gives the impression that she is passing on her own practices and her own forms of knowledge. Her main resource seems to be her own experience of using this algorithm.

### **Transformation**

Laura's own ability to perform column multiplication is secure, but her pedagogical challenge is to *transform* what she knows for herself into a form that can be accessed and appropriated by the children. Laura's choice of demonstration examples in her introduction to column multiplication merits some consideration and comment. Her first example is  $37 \times 9$ ; she then goes on to work through  $49 \times 8$  and  $19 \times 4$ . Now, the NNS emphasises the importance of *mental* methods, where possible, and also the importance of choosing the most suitable strategy for any particular calculation.  $49 \times 8$  and  $19 \times 4$  can all be more efficiently performed by rounding up, multiplication and compensation e.g.  $49 \times 8 = (50 \times 8) - 8$ .

Her choice of exercises - the practice examples - also invites some comment. The sequence is:  $19 \times 4$ ,  $27 \times 9$ ,  $42 \times 4$ ,  $23 \times 6$ ,  $37 \times 5$ ,  $54 \times 4$ ,  $63 \times 7$ ,  $93 \times 6$ , with  $99 \times 9$ ,  $88 \times 3$ ,  $76 \times 8$ ,  $62 \times 43$ ,  $55 \times 92$ ,  $42 \times 15$  as 'extension' exercises (although no child actually attempts these in the lesson). Our earlier remark about the suitability of the column algorithm relative to alternative mental strategies applies to several of these,  $99 \times 9$  being a notable example.

But suppose for the moment that it is understood and accepted by the pupils that they are to put aside consideration of such alternative strategies - that these exercises are there merely as a vehicle for them to gain fluency with the algorithm. In that case, the *sequence* of exercises might be expected to be designed to present the pupils with increasing challenge as they progress through them. This challenge would be of two kinds. First, the partial products (e.g.  $10 \times 4$  and  $9 \times 4$ ) make demands on their recall of products, but consideration of this dimension is not apparent in the sequence. Secondly, the necessity of 'carrying' when summing the *tens* digits of the partial products would add to the complexity of an exercise. This is a factor only in the second of the first eight exercises,  $27 \times 9$ , and in the third of the extension items,  $76 \times 8$ .

**Discussion point:** on what grounds did Laura choose these particular examples and exercises? What considerations might contribute to the choice?

### Connection

Perhaps the most immediate connection to be established in this lesson is that between the grid method and the column algorithm. Laura seems to have this connection in mind as she introduces the main activity. She reminds them that they have used the grid method, and says that she will show them a “slightly different way of writing it down”, although after the first example is completed Laura says that they are learning a “different way to work it out”. She says that the answer would be the same whichever way they did it “because it’s the same sum”. Of course, that presupposes that both methods are valid, but does not clarify the connection between them: that the same processes and principles - partition, distributivity and addition - are present in both methods. The fact that Laura includes demonstrations of  $37 \times 9$  by both methods does help to establish the connection, but the effort to sustain the connection is not maintained, and no reference to the grid method is made in her second demonstration example,  $49 \times 8$ . Her presentation of this example now homes in on procedural aspects - the need to “partition the number down”, “adding a zero” to  $8 \times 4$ , getting the columns lined up, adding the partial products from the right. The fact that the connection is tenuous for at least one pupil is apparent in the plenary. Sean actually volunteers to calculate  $27 \times 9$  on the whiteboard. He writes 27 and  $\times 9$  in the first two rows as expected, but then writes  $20 \times 7$  and  $2 \times 9$  to the left in the rows below.

**Discussion point:** Laura is clearly trying to make a connection between the grid method and the column method. What reasons did she have in mind for doing so? To what extent did she think she was successful?

### Contingency

Sean’s faulty attempt (mentioned above) to calculate  $27 \times 9$  on the whiteboard appears to have taken Laura by surprise. Laura did not notice Sean’s error immediately - it seems that she fully expected him to apply the algorithm faultlessly, and that his actual response really was unanticipated. In the event, there are several ‘bugs’ in his application of the procedure. The partition of 27 into 20 and 7 is faulty, and the multiplicand is first 9, then 7. This would seem to be a case where Sean might be encouraged to reconsider what he has written by asking him some well-chosen questions. One such question might ask how he would do it by the grid method. Or simply why he wrote those particular numbers where he did. Laura avoids correcting Sean herself, although her response indirectly suggests that all is not well. She asks the class? “Is that the way to do it? Would everyone do it that

way?”. Leroy demonstrates the algorithm correctly, but there is no diagnosis of where Sean went wrong, or why.

**Discussion point:** what might be the reason for Sean’s error? In what ways could this have been addressed in the lesson, or subsequently?

## **APPLICATIONS AND NEXT STEPS**

Research originally fuelled by curiosity about teacher knowledge and classroom practices led to the development of a tool for ‘seeing’ and analysing mathematics teaching: the Knowledge Quartet. In this paper, we have introduced the Quartet, and shown its relevance and usefulness in our analysis of part of Laura’s lesson with a Year 5 class. We have a manageable framework within which to discuss actual, observed teaching sessions with trainees and their mentors. These groups of participants in initial teacher preparation, as well as our university-based colleagues, need to be acquainted with (and convinced of the value of) the Quartet, and to be familiar with some details of its conceptualisation, as described in this paper. Within the last two years we have taken steps towards this familiarisation in the context of our own university’s pre-service elementary and middle school teacher education programmes. The four dimensions of the Knowledge Quartet have been used as a framework for lesson observation and reflection. Initial indications are that this development has been well received by mentors, who appreciate the specific focus on *mathematics* content and pedagogy. They observe that it compares favourably with guidance on mathematics lesson observation from the NNS itself, which focuses on more generic issues such as “a crisp start, a well-planned middle and a rounded end. Time is used well. The teacher keeps up a suitable pace and spends very little time on class organisation, administration and control.” (DfEE, 2000, p.11). It is all too easy for analysis of a lesson taught by a novice teacher to be (or be perceived to be) gratuitously critical, and we therefore emphasise that the quartet is intended as a tool to *support* teacher development, with a sharp and structured focus on the impact of their SMK and PCK on teaching. Indications of how this might work are explicit in our analysis of Laura’s lesson. We have emphasised that our analysis has been *selective*: we raised for attention some issues, but there were others which, not least out of space considerations, we chose not to mention. The same would be likely to be true of the review meeting - in that case due to *time* constraints, but also to avoid overloading the trainee with action points. Such a meeting might well focus on a lesson fragment, and on only one or two dimensions of the knowledge quartet for similar reasons. At our university, mentor training on the knowledge quartet and its use has emphasised the need to be specific and selective in the use of feedback. Mentors took part in workshops in which several groups observed

the same videotaped lesson with a focus on one of the four dimensions of the quartet. Useful discussions followed concerning the quantity and type of feedback that would be appropriate. Any tendency to descend into deficit discourse is also tempered by consideration of the wider context of the student teacher's experience in school. In the novice teacher we see the very beginnings of a process of reconciliation of pre-existing beliefs, new 'theoretical' knowledge, 'practical' advice received from various quarters, in the context of highly-pressured, high-stakes school-based placements. There is also good evidence (e.g. Hollingsworth, 1988; Brown, Mcnamara, Jones, and Hanley, 1999) that trainees' concern for pupil learning is often eclipsed by their anxieties about timing, class management and pupil behaviour. In an attempt to recognise and address this, the Knowledge Quartet is being introduced to trainee teachers at our university, in order to direct attention to the subject content dimension of their classroom practice, and the ways that content knowledge might inform their planning, preparation and teaching. These sessions have also been well-received, with the trainees welcoming the Quartet as a way of framing their thinking about their teaching. Many expressed interest in participating in Quartet-related research in their first teaching appointment (see below). By introducing mentors, mathematics co-ordinators and trainees to the knowledge quartet, we provide all relevant 'stakeholders' with a framework for the discussion of mathematics planning and teaching that will encourage a focus on the subject content as well as the management of lessons. In addition, we have recently been analysing lessons taught by secondary mathematics PGCE students, through the lens of the Knowledge Quartet. We could add that colleagues working in English and in science education see potential in the Quartet for their own lesson observations and review meetings: it would be interesting to see what the conceptualisations of the dimensions of the Quartet might look like in these and other subject disciplines. A more fundamental question arises, however: can a framework for knowledge-in-teaching developed in one subject discipline be legitimately adopted in another?

Our research on the application of the Knowledge Quartet as a tool to support teacher development is currently being extended within a project working with beginning teachers during their initial training and the first three years of their teaching (see e.g. Turner, 2006, 2008). This project grows out of a recognition that the mathematical knowledge and understandings of teachers, their beliefs about the nature of mathematics and the teaching and learning of mathematics, cannot be radically changed within the one-year PGCE. In this project the participants will be helped to become increasingly familiar with the quartet in order to use it as a shared language for discussion of, and reflection on, their mathematics teaching. Trainees from one cohort of the Early Years and Primary PGCE course are working worked with a researcher and discussing video-tapes of their lessons with

specific reference to the four dimensions of the Knowledge Quartet. This is a four year longitudinal study looking at how the Quartet may be used with beginning teachers and their mentors to develop mathematics teaching. It is also expected that findings from this study will inform the development of the conceptualisation of the Quartet and our understanding of how to facilitate its use by others.

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