

## CATS, Count&Add in Time&Space - a Natural Way to Become a Mathematics Teacher

Allan Tarp, the MATHeCADEMY.net, Allan.Tarp@gmail.com

Hornslets Alle 27, DK8500 Grenaa Denmark, +45 8632 1899

*The CATS-approach, Count&Add in Time&Space, is a natural way to become a math teacher. It obeys the fundamental rule of good research, never to ask leading questions. To learn mathematics, students should not be taught mathematics; instead they should meet the roots of mathematics, multiplicity. Through guiding educational questions asking them to Count and Add in Time and Space, they learn mathematics without knowing it. The CATS-approach is rich on examples of recognition and new cognition to be observed, reflected and reported by teachers and researchers.*

### The Background

Enlightenment mathematics treated mathematics as a natural science. Grounded in the natural fact many, it induced its definitions as abstractions from examples, and validated its statements by testing deductions on examples (Kline 1972: 398). Inspired by the invention of the set-concept, modern mathematics turned Enlightenment mathematics into a purely deductive 'metamatics' that by defining its concepts as examples of abstractions, and by proving its statements as deductions from meta-physical axioms, needs no outside world and becomes entirely self-referring. However, a self-referring mathematics soon turned out to be an impossible dream. With his paradox about the set of sets not being a member of itself, Russell proved that using sets implies self-reference and self-contradiction known from the classical liar-paradox 'this statement is false' being false when true and true when false: definition  $M = \{ A \mid A \notin A \}$ , statement  $M \in M \Leftrightarrow M \notin M$ .

Likewise, without using self-reference it is impossible to prove that a proof is a proof; a proof must be defined. And Gödel soon showed that theories couldn't be proven consistent since they will always contain statements that can neither be proved nor disproved.

Without an alternative, modern mathematics creates big problems to mathematics education, as e.g. the worldwide enrolment and justification problems in mathematical based educations and teacher education (Jensen et al 1998); and 'the relevance paradox formed by the simultaneous objective relevance and subjective irrelevance of mathematics' (Niss in Biehler et al 1994: 371).

And teachers are forced to bring meaningless modern mathematics to the everyday classroom:

When insisting that general functions must be taught before linear and exponential functions that must be taught before teaching change by adding or multiplying a constant number. Students have no problems seeing that the change  $200\$ + 5\$ \times$  times can be generalized to  $y = b + a \cdot x$ . But they refuse to learn that 'a functions is an example of a many-one set-relation', which they hear as 'bublibub is an example of bablibab'. A rational reaction since defining a 1700-concept as an example of a more abstract 1900-concept is just another example of 'metamatics'.

And when teaching that fractions can be added without units, thus transforming grounded mathematics into ungrounded ‘mathematism’ true in a library but not in a laboratory:

The teacher:	The students:
Welcome! What is $1/2 + 2/3$ ?	$1/2 + 2/3 = (1+2)/(2+3) = 3/5$
No, $1/2 + 2/3 = 3/6 + 4/6 = 7/6$	But $1/2$ of 2 cokes + $2/3$ of 3 cokes is $3/5$ of 5 cokes! How can it be 7 cokes out of 6 cokes?
Inside this classroom $1/2 + 2/3 = 7/6!$	

So in reality, what is called ‘mathematics education’ is not education in mathematics, but instead education in ‘metamatism’ merging metamatics with mathematism. To design an alternative, mathematics should return to its roots, multiplicity, guided by a kind of research able at uncovering hidden alternatives to choices presented as nature, as recommended by the ancient Greek sophists.

### **Anti-pastoral Sophist Research**

Ancient Greece saw a struggle between the sophists and the philosophers as to the nature of knowledge. The sophists warned that to protect democracy people should be enlightened to tell choice from nature in order to prevent patronization presenting its choices as nature. To the philosophers, seeing everything physical as examples of meta-physical forms only visible to them, patronization was a natural order when left to the philosophers (Russell 1945).

The Greek democracy vanished with the Greek silver bringing wealth by financing trade with Far-East luxury goods as silk and spices. Later this trade was reopened by German silver financing the Italian Renaissance; and by silver found in America. Robbing the slow Spanish silver ships returning on the Atlantic was no problem to the English; finding a route to India on open sea was. Until Newton found out that when the moon falls to the earth as does the apple, it is not obeying the unpredictable will of a meta-physical patronizor only attainable through faith, praying and church attendance; instead it is following its own predictable physical will attainable through knowledge, calculations and school attendance.

This insight created the Enlightenment period: when an apple obeys its own will, people should do the same and replace patronization with democracy. Two democracies were installed, one in the US, and one in France. The US still has its first republic; France now has its fifth.

The German autocracy tried to stop the French democracy by sending in an army. However, the German mercenaries were no matches to the French conscripts only too aware of the feudal consequences of loosing. So the French stopped the Germans, and later occupied Germany.

Unable to use the army, the German autocracy used the school to stop the enlightenment in spreading from France. Humboldt was asked to create an elite school, and used Bildung as counter-enlightenment to create the self-referring Humboldt University (Denzin et al 2000: 85).

Inside the EU the sophist warning is kept alive in the French postmodern or post-structural thinking of Derrida, Lyotard and Foucault warning against patronizing categories, discourses and institutions presenting their choices as nature (Tarp 2004).

Derrida recommends that patronizing categories, called logocentrism, be ‘deconstructed’:

Derrida encourages us to be especially wary of the notion of the centre. We cannot get by without a concept of the centre, perhaps, but if one were looking for a single ‘central idea’ for Derrida’s work it might be that of decentring. It is in this very general context that we might situate the significance of ‘poststructuralism’ and ‘deconstruction’: in other words, in terms of a decentring, starting with a decentring of the human subject, a decentring of institutions, a decentring of the logos. (Logos is ancient Greek for ‘word’, with all its connotations of the authority of ‘truth’, ‘meaning’, etc.) (..) It is a question of the deconstruction of logocentrism, then, in other words of ‘the centrism of language in general’. (Royle 2003: 15-16)

As to discourses Lyotard coins the term ‘postmodern’ when describing ‘the crisis of narratives’:

I will use the term modern to designate any science that legitimates itself with reference to a metadiscourse (..) making an explicit appeal to some grand narrative (..) Simplifying to the extreme, I define postmodern as incredulity towards meta-narratives. (Lyotard 1984: xxiii, xxiv)

Foucault calls institutional patronization for ‘pastoral power’:

The modern Western state has integrated in a new political shape, an old power technique which originated in Christian institutions. We call this power technique the pastoral power. (..) It was no longer a question of leading people to their salvation in the next world, but rather ensuring it in this world. And in this context, the word salvation takes on different meanings: health, well-being (..) And this implies that power of pastoral type, which over centuries (..) had been linked to a defined religious institution, suddenly spread out into the whole social body; it found support in a multitude of institutions (..) those of the family, medicine, psychiatry, education, and employers. (Foucault in Dreyfus et al 1982: 213, 215)

In this way Foucault opens our eyes to the salvation promise of the generalized church: ‘you are un-saved, un-educated, un-social, un-healthy! But do not fear, for we the saved, educated, social, healthy will save you. All you have to do is: repent and come to our institution, i.e. the church, the school, the correction center, the hospital, and accept becoming a docile lackey’.

To Foucault, institutions building on discourses building on categories build upon choice, so they all have a history, a ‘genealogy’, that can be uncovered by ‘knowledge archeology’.

The French skepticism towards words, our most fundamental institution, is validated by a ‘number&word observation’: Placed between a ruler and a dictionary a so-called ‘17 cm long stick’ can point to ‘15’, but not to ‘stick’; thus it can itself falsify its number but not its word, which makes numbers nature and words choices becoming pastoral if hiding their alternatives, and allows number-statements to be research, whereas word-statements will always be interpretations.

On the basis of the ancient and the contemporary sophist warning, a research paradigm can be created called ‘anti-pastoral sophist research’ deconstructing pastoral choices presented as nature by uncovering hidden alternatives. Thus anti-pastoral sophist research doesn’t refer to but deconstruct existing research by asking ‘in this case, what is nature and what is pastoral choice presented as nature, thus covering alternatives to be uncovered by anti-pastoral sophist research?’

## Natural Learning and Natural Research

To make categories, discourses and institutions not pastoral but enlightening, they should be grounded, not in choice but in nature, by using Grounded Theory, the method of natural research developed in the other Enlightenment democracy, the American; and resonating with Piaget's principles of natural learning (Piaget 1970), and with the Enlightenment principles for research: Observe, abstract categories and relations to be accommodated through predicted deductions.

Grounded theory also shows skepticism towards existing research-based categories:

Although categories can be borrowed from existing theory, provided that the data are continually studied to make certain that the categories fit, generating theory does put a premium on emergent conceptualizations. There are a number of reasons for this. Merely selecting data for a category that has been established by another theory tends to hinder the generation of new categories, because the major effort is not generation, but data selection. Also, emergent categories usually prove to be the most relevant and the best fitted to the data. As they are emerging, their fullest possible generality and meaning are continually being developed and checked for relevance. Also the adequacy of indicators for emergent categories is seldom a problem. (...) In short, our focus on the emergence of categories solves the problems of fit, relevance, forcing, and richness. An effective strategy is, at first, literally to ignore the literature of theory and fact on the area under study, in order to assure that the emergence of categories will not be contaminated by concepts more suited to different areas. Similarities and convergences with the literature can be established after the analytic core of categories has emerged. (Glaser et al 1967: 36-37)

### A Historical Background

The natural fact many provoked the creation of mathematics as a natural science addressing the two fundamental human questions 'how to divide the earth and what it produces?'

Distinguishing the different degrees of many leads to counting that leads to numbers.

1.order counting counts in 1s and creates number-icons by rearranging the sticks so that there are five sticks in the five-icon 5 etc. if written in a less sloppy way.

2.order counting counts by bundling&stacking in numbers with a name and an icon, resulting in a double stack of bundled and unbundled, e.g.  $T = 3 \text{ 5s} + 2 \text{ 1s} = 3)2)$  if using cup-writing, leading to decimal-writing separating the left bundle-cup from the right single-cup:  $T = 3)2) = 3.2 \text{ 5s} = 3.2 * 5$ .

The result can be predicted by the 'recount-formula'  $T = (T/b) * b$  iconizing that counting in bs means taking away bs  $T/b$  times, e.g.  $T = (4*5)/7*7 = 2*7 + 6*1 = 2)6) = 2.6*7$ .

3.order counting counts in tens, having a name but not an icon since the bundle-icon is never used: counting in 5s,  $T = 5 \text{ 1s} = 1 \text{ 5s} = 1.0 \text{ bundle} = 10$  if leaving out the decimal and the unit.

In Greek, mathematics means knowledge, i.e. what can be used to predict with, making mathematics a language for number-prediction: The calculation '2+3 = 5' predicts that counting-on 3 times from 2 will give 5. '2\*3 = 6' predicts that repeating adding 2 3 times will give 6. '2^3 = 8' predicts that repeating multiplying with 2 3 times will give 8. Also, any calculation can be turned around and become a reversed calculation predicted by the reversed operation: In the question '3+x = 7' the answer is predicted by the calculation  $x = 7-3$ , etc.

$3+x = 7$ $x = 7-3$	$3*x = 7$ $x = 7/3$	$x^3 = 7$ $x = \mathbf{Error!}$ <b>Bookmark not defined.</b> $\sqrt[3]{7}$	$3^x = 7$ $x = \log_3(7)$
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Thus the natural way to solve an equation is to move a number across the equation sign from the left forward-calculation to the right backward-calculation side, reversing its calculation sign.

In Arabic, algebra means reuniting, i.e. splitting a total in parts and (re)uniting parts into a total. The operations + and \* unite variable and constant unit-numbers; ∫ and ^ unite variable and constant per-numbers. The inverse operations – and / split a total into variable and constant unit-numbers; d/dx and √ & log split a total into variable and constant per-numbers:

Totals unite/split into	Variable	Constant
Unit-numbers \$, m, s, ...	$T = a + n$ $T - n = a$	$T = a * n$ $T/b = a$
Per-numbers \$/m, m/s, m/100m = %, ...	$\Delta T = \int f dx$ $dT/dx = f$	$T = a ^ n$ $\sqrt[n]{T} = a$ , $\log_a T = n$

In Greek, geometry means earth measuring. Earth is measured by being divided into triangles, again being divided into right-angled triangles, each seen as a rectangle halved by a diagonal.

Recounting the height h and base b in the diagonal d produces three per-numbers:

$$\sin A = \text{height/diagonal} = h/d, \tan A = \text{height/base} = h/b, \cos A = \text{base/diagonal} = b/d.$$

However, needing the Arabic numbers, Greek geometry turned into Euclidean geometry, freezing the development of mathematics until the Enlightenment century:

The enthusiasm of the mathematicians was almost unbounded. They had glimpses of a promised land and were eager to push forward. They were, moreover, able to work in an atmosphere far more suitable for creation than at any time since 300 B.C. Classical Greek geometry had not only imposed restrictions on the domain of mathematics but had impressed a level of rigor for acceptable mathematics that hampered creativity. The seventeen-century men had broken both of these bonds. Progress in mathematics almost demands a complete disregard of logical scruples; and, fortunately, the mathematicians now dared to place their confidence in intuitions and physical insights. (Kline 1972: 399)

The success was so overwhelming that mathematicians feared that mathematics, called geometry at that time, had come to a standstill at then end of the 18<sup>th</sup> century:

Physics and chemistry now offer the most brilliant riches and easier exploitation; also our century's taste appears to be entirely in this direction and it is not impossible that the chairs of geometry in the Academy will one day become what the chairs of Arabic presently are in the universities. (Lagrange in Kline 1972: 623)

But in spite of the fact that calculus and its applications had been developed without it, logical scruples soon were reintroduced arguing that both calculus and the real numbers needed a rigorous foundation. So in the 1870s the concept 'set' reintroduced rigor into mathematics, leading to

modern set-based mathematics dreaming that it could become a collection of well-proven statements about well-defined concepts; a dream that broke with Russell and Gödel.

### **Where do concepts come from**

The number one question in education is ‘where do concepts come from - from above or from below - from the outside or from the inside?’ The four different answers to this question lead to four different ‘learning rooms’.

The traditional learning room is the ‘transmitter room’ saying ‘from above&outside’ and seeing knowledge as outside information to be transmitted into learners’ brains with teachers and textbooks as channels. General communication theory is applied in this learning room.

Its traditional alternative is the ‘constructivist room’ saying and ‘from above&inside’ and seeing knowledge as library information that can be reached by learners through internal scaffolding guided by a teacher. Vygotsky and Bruner constructivism is applied in this learning room.

One hidden alternative is the ‘story-telling room’ saying ‘from below&outside’ and seeing knowledge as unknown stories about known subjects that the learner, biologically programmed to remember gossip, automatically learns if told as gossip by a story-teller, a teacher. Story-telling and fairy tales are applied in this learning room.

Another hidden alternative is the ‘apprentice room’ saying ‘from below&inside’ and seeing knowledge as both verbal and tacit that the learner, biologically programmed to adapt to the surroundings through assimilation and accommodation, automatically learns if placed in exemplary situations by a teacher. Piaget constructivism and Situated Learning is applied in this learning room.

The traditional learning rooms take mathematics for granted and see the world as applying mathematics. The hidden learning rooms have the opposite view taking the multiplicity of the outside world for granted and as a creator of mathematics through the principle ‘grip to grasp’, ‘through the hand to the head’ or ‘greifen vor begreifen’.

The transmission room versus the story-telling room arranges sentence-loaded educational meetings with sentences with abstract versus concrete subjects. The constructivist room versus the apprentice room arranges sentence-free educational meetings with abstract versus concrete subjects.

Thus the two ‘from above’ rooms, seeing mathematics as book-based knowledge, want learning to take place in a library; and the two ‘from below’ rooms, seeing mathematics as a natural science investigating the natural fact many, want learning to take place in a laboratory.

Humans develop from one biological species to another. Before puberty human children learn about the outside world by adapting, i.e. through feeling and experimenting, as mammal offspring do. Of course, human language makes human learning far more effective than other mammal learning. After puberty mammals stop learning and focus on feeding and guarding the offspring

while adapting. Humans, however, have a language that can be used to communicate information between brains, most effectively in the form of gossip about known subjects.

### Word-language and Number-language

A ruler and a dictionary help us to assign numbers and words to things using our number-language and our word-language. Thus we have word-sentences containing a subject, a verb, and an object; and we have number-sentences, equations, containing a quantity, an equation sign, and numbers and calculations. Both sentences are describing the world, and both are being described by a meta-language. The meta-language of the word-language is called grammar. The meta-language of the number-language is called mathematics.

Our two languages and their meta-languages constitute a language-house with two floors. In the lower floor the language is used to describe the world, and in the upper floor the meta-language is used to describe the language. Syntax errors occur if the meta-language is used to describe the world: ‘the verb got drunk’. So mathematics does not describe the world, mathematics describes the number-language, and the number-language describes the world.

		<b>THE LANGUAGE HOUSE</b>		
META-LANGUAGE	Grammar	Subject	Constants and variables	Mathematics
LANGUAGE	Word-language	<i>The pencil is red</i>	<i>Area = length*height</i>	Number-language
WORLD	THINGS IN TIME AND SPACE			

### Constructing an Enlightening Mathematics From Below

Enlightening mathematics sees mathematics as a natural science investigating the natural fact many. Thus it respects two fundamental principles:

- a Kronecker-principle saying that only the natural numbers can be taken for granted; and
- a Russell-principle saying that we cannot use self-reference and talk about sets of sets.

The enlightening mathematics at the website MATHeCADEMY.net has been constructed as such an example of a Kronecker-Russell mathematics based on studying multiplicity in a set-free ‘Count&Add laboratory’ where addition predicts counting-results, thus making mathematics a language for number-prediction.

The website contains 2\*4 study units in ‘mathematics from below, the LIB-free LAB-approach’, organized as laboratory activities where the learner learns ‘CATS’, i.e. to Count & Add in Time & Space guided by educational questions Q and answers A.

The study units CATS1 are for primary school and the study units CATS2 are for secondary school. The units were developed for a web-based distance-learning course in mathematics at a Danish teacher college.

### Counting C1

**Q:** How to count multiplicity? **A:** By bundling and stacking the total  $T$  predicted by  $T = (T/b)*b$

**Q:** How to recount 8 in 3s:  $T = 8 = ? 3s = ?*3$ . **A:**  $8 = (8/3)*3 = 2*3 + 2*1 = 2)2) = 2.2*3 = 2 \frac{2}{3}*3$

**Q:** How to recount 6kg in \$:  $T = 6kg = ?\$$ . **A:** If  $4kg = 2\$$  then  $6kg = (6/4)*4kg = (6/4)*2\$ = 3\$$

**Q:** How to count in standard bundles? **A:** Bundling bundles gives a multiple stack, a stock or polynomial:  $T = 423 = 4\text{BundleBundle} + 2\text{Bundle} + 3 = 4\text{tente}2\text{ten}3 = 4*B^2 + 2*B + 3$

Section C1 looks at ways to count multiplicity. Spatial multiplicity is representing temporal repetition through sticks and strokes. A multiplicity of sticks can be rearranged in icons so that there are four sticks in the icon 4 etc. Then a given total  $T$  can be counted in e.g. 4s by repeating the process ‘from  $T$  take away 4’, which can be iconized as ‘ $T-4$ ’; where the repeated process ‘from  $T$  take away 4s’ can be iconized as ‘ $T/4$ ’. This makes it possible to predict the counting-result through a calculation using the ‘recount-equation’  $T = (T/b)*b$ . Leftovers are stacked as 1s creating a stock  $T = 2*3 + 2*1$ . The stacks can be placed in two cups, a left bundle-cup and a right single-cup, and described by cup-writing  $T = 2)2)$ , or decimal-writing including the unit  $T = 2)2) = 2.2 3s = 2.2*3$ ; or the leftovers can be counted in 3s and added on top of the 3-stack:  $T = 2 \frac{2}{3} * 3 = 2 \frac{2}{3} 3s$ .

Changing units is another example of a recounting where a given total is double-counted in two different units e.g.  $T = 4\$ = 5kg$  producing a per-number  $4\$/5kg = 4/5 \$/kg$ . Thus to answer the question ‘ $7kg = ?\$$ ’ we just have to recount the 7 in 5s:  $T = 7kg = (7/5)*5kg = (7/5)*4\$ = 5 \frac{3}{5}\$$ .

The number ten has a name but no icon, since the bundle-size is not used: Counting in 5s,  $5 1s = 1 5s = 1$  bundle. Before introducing ten as the standard-bundle and leaving out the units,  $2.4$  tens = 24, the core of mathematics can be learned by using 1 digit numbers alone. (Zybartas et al 2005).

### Adding A1

**Q:** How to add stacks concretely?  $T = 27 + 16 = 2 \text{ ten } 7 + 1 \text{ ten } 6 = 3 \text{ ten } 13 = ?$ .

**A:** By restacking overloads predicted by the ‘restack-equation’  $T = (T-b) + b$ :

$T = 27 + 16 = 2 \text{ ten } 7 + 1 \text{ ten } 6 = 3 \text{ ten } 13 = 3 \text{ ten } 1 \text{ ten } 3 = 4 \text{ ten } 3 = 43$ .

**Q:** How to add stacks abstractly? **A:** Vertical addition uses carrying. Horizontal addition uses FOIL.

Section A1 looks at how stacks can be added by removing overloads that often appears when one stack is placed on top of another stack. The overload leads to ‘internal trade’ between two stacks where a stack of 10 1s is rebundled and restacked as 1 10-bundle. The result can be predicted by a calculation on paper using either a vertical way of writing the stacks using carrying to symbolize the

internal trade; or using a horizontal way of writing the stacks using the FOIL-principle (First, Outside, Inside, Last). In both cases the overload can be restacked predicted by the restack-equation  $(T-b) + b$ , and recounted predicted by the recount-equation  $T = (T/b)*b$ .

### Time T1

**Q:** How can counting & adding be reversed? **A:** By calculating backwards, i.e. by moving a number to the other side of the equation sign and reversing its calculation sign.

**Q:** Counting 3s and adding 2 gave 14. **A:**  $x * 3 + 2 = 14$  is reversed to  $x = (14 - 2)/3$ .

**Q:** Can all calculations be reversed? **A:** Yes.  $x + a = b$  is reversed to  $x = b - a$ ,  $x * a = b$  is reversed to  $x = b/a$ ,  $x^a = b$  is reversed to  $x = \sqrt[a]{b}$ ,  $a^x = b$  is reversed to  $x = \log_b/a$ .

Section T1 looks at formulas, the sentences of the number-prediction language. Containing two unknown variables, a formula becomes a function to be tabled and graphed. Containing one unknown variable, a formula becomes an equation to be solved by reversing the calculations, moving numbers from the forward-calculation side to the backward-calculation side reversing their signs:  $x*3+2 = 14$  is reversed to  $x = (14-2)/3$ . This forward and backward calculation method gives a new perspective on the classical quantitative literature consisting of word-problems.

### Space S1

**Q:** How to count the plane and spatial properties of stacks, boxes and round objects?

**A:** By using a ruler, a protractor and a triangular shape; by the 3 Greek Pythagoras', mini, midi & maxi; and by the 3 Arabic recount-equations:  $\sin A = a/c$ ,  $\cos A = b/c$ ,  $\tan A = a/b$ .

Section S1 looks at how to describe plane properties of stacks as areas and diagonals by the 3 Greek Pythagoras', mini, midi & maxi; and by the 3 Arabic recount-equations:  $\sin A = a/c$ ,  $\cos A = b/c$  and  $\tan A = a/b$ . A circle can be divided into many right-angled triangles whose heights add up to the circumference  $C$  of the circle:  $C = 2 * r * (n*\sin(180/n)) = 2 * r * \pi$  for  $n$  sufficiently big.

Finally we look at how to describe spatial properties of solids such as surfaces and volumes by formulas and by a 2-dimensional representation of 3-dimensional shapes.

### Counting C2

**Q:** How to count possibilities? **A:** By using the numbers in Pascal's triangle.

**Q:** How to predict unpredictable numbers? **A:** If a 'post-diction' gives the average 8.2 with deviation 2.3, the 'pre-diction' gives the confidence interval  $8.2 \pm 2*2.3$  with 95% probability.

Section C2 looks at numbers that change unpredictably as e.g. in surveys. Through counting we can set up a frequency-table accounting for the previous behavior of the numbers. From this table their average level and their average change can be calculated. From this we can predict that with a 95%

probability, future numbers will occur within an interval determined by the average level and double the average change. Counting the numbers of wins when repeating a game with winning probability  $p$  is another example of an unpredictable number, also called a stochastic variable.

### Adding A2

**Q:** What is a per-number? **A:** Per-numbers occur when counting, when pricing and when splitting.

**Q:** How to add per-numbers? **A:** The \$/day-number  $a$  is multiplied with the day-number  $b$  before added to the total \$-number  $T$ :  $T_2 = T_1 + a*b$

Section A2 looks at how to add per-numbers by transforming them to totals. The \$/day-number  $a$  is multiplied with the day-number  $b$  before added to the total \$-number  $T$ :  $T_2 = T_1 + a*b$ . 2days at 6\$/day + 3days at 8\$/day = 5days at 7.2\$/day. And 1/2 of 2 cans + 2/3 of 3 cans = 3 of 5 cans = 3/5 of 5 cans. Repeated and reversed addition of per-numbers leads to integration and differentiation:

$$T_2 = T_1 + a*b; T_2 - T_1 = a*b; \Delta T = \sum a*b = \int y*dx; \text{ and}$$

$$T_2 = T_1 + a*b; T_2 - T_1 = a*b; a = (T_2 - T_1)/b = \Delta T/\Delta b = dy/dx$$

### Time T2

**Q:** How to predict the terminal number when the change is constant?

**A:** By constant change-equations: If  $K_0 = 30$  and  $\Delta K/n = a = 2$ , then  $K_7 = K_0 + a*n = 30 + 2*7 = 44$ .

$$\text{If } K_0 = 30 \text{ and } \Delta K/K = r = 2\%, \text{ then } K_7 = K_0*(1+r)^n = 30*1.02^7 = 34.46$$

**Q:** How to predict the terminal number when the change is variable, but predictable?

**A:** By a variable change-equation: If  $K_0 = 30$  and  $dK/dx = K'$ , then  $K - 30 = \Delta K = \int dK = \int K' dx$

Section T2 looks at how a stack changes in time by adding a constant number, or by a constant percent where adding 5% means changing 100% to 105%, i.e. multiplying with 105% = 1.05.

If related by a formula  $y = f(x)$ , a  $x$ -change  $\Delta x$  will effect a  $y$ -change  $\Delta y$  that can be recounted in the  $x$ -change as  $\Delta y = (\Delta y/\Delta x)*\Delta x$ , or  $dy = (dy/dx)*dx = y'*dx$  in the case of micro-changes.

If a stack  $y$  changes by adding variable predictable numbers  $dy$ , summing up the single  $y$ -changes gives the total  $y$ -change, i.e. the terminal  $y_2$  minus the initial  $y_1$ :  $\int dy = \int y'*dx = y_2 - y_1$ .

### Space S2

**Q:** How to predict the position of points and lines? **A:** By using a coordinate-system:

$$\text{If } P_0(x,y) = (3,4) \text{ and } \Delta y/\Delta x = 2, \text{ then } P_1(8,y) = P_1(x+\Delta x, y+\Delta y) = P_1(3+(8-3), 4+2*(8-3)) = (8,14)$$

Section S2 looks at how to predict the position of points and lines and geometrical figures and graphs using a coordinate system. Then we look at how to use the new calculation technology such as computers and calculators to calculate a set of numbers, vectors, and a set of vectors, matrices.

## Quantitative Literature

**Q:** What is quantitative literature? **A:** Quantitative literature is about multiplicity in time and space.

**Q:** Does quantitative literature share the 3 different genres: fact, fiction and fiddle? **A:** Yes.

In formulas as  $T = c \cdot p$  we need to know what quantities are described to determine the truth-value of the formula's prediction. It turns out that both word-statements and number-statements share the same genres: fact, fiction and fiddle. Fact-models predict predictable quantities. Fiction-models predict unpredictable quantities. Fiddle-models predict qualities.

### Modelling With Regression-Mathematics

Modelling consists of four parts: a real-world problem, a model problem, a model solution, and a real-world solution. First a real-world problem leads to a model problem, often a table relating two variables  $x$  and  $y$ . Then in the model solution, regression is used to find a formula connecting the variables. Containing one unknown, the formula becomes an equation that can be solved manually or by the Math Solver on a Graphical Display Calculator. Containing two unknowns, the formula becomes a function to be illustrated by a graph; and where the two typical questions 'given  $x$  find  $y$ ' and 'given  $y$  find  $x$ ' reduces the function to an equation to be solved by the Math Solver, or graphically by Trace and by Calc Intersection. Finally the solution is evaluated as to its applicability as a real-world solution. Computers have enabled the creation of huge models using system dynamics to model the interaction of complex systems of variables in science and in economics.

### Fact Models

Fact models quantify and predict predictable quantities: 'What is the area of the walls in this room?' In fact models the predicted answer is what is observed. Hence numbers calculated by a fact model can be trusted. Geometry and the basic formulas  $T = 2+3$ ,  $T = 2 \cdot 3$  and  $T = 2^3$  etc. are fact models, as well as many models from physics, trade and financing. A fact model may also be called a 'since-then model' or a 'room-model'.

### Fiction Models

Fiction models quantify and predict unpredictable quantities: 'My debt will soon be paid off at this rate!' Fiction models produce predictions based upon presumed assumptions that should be supplemented with alternative assumptions, i.e. with parallel scenarios. Typical examples are average-models, simplifying complex economical or technical models by assuming some variables to be constants staying at their average level. Other examples are linear demand and supply curves in economical theory. A fiction model may also be called an 'if-then' or a 'rate-model'.

### Fiddle Models

Fiddle models quantify qualities: 'Are the risk and casualty numbers of this road high enough to cost a bridge?' This question will install crosswalks instead of bridges on motorways since it is

cheaper to be in a cemetery than at a hospital. Fiddle models should be rejected asking for a word description instead. Many risk-models are fiddle models. The basic risk model says: Risk = Consequence \* Probability. A fiddle model may also be called a 'so-what' model or a 'risk-model'.

### **The Grand Narratives of the Quantitative Literature**

Literature is narratives about real-world persons, actions and phenomena. Quantitative literature also has its grand narratives. That overwhelmingly many numbers can be added by one simple difference, providing the numbers can be written as change-numbers, is a grand narrative.

Another example is the geometry of a tennis ball, forcing much traditional geometry to be adapted: parallel lines intersect, the angles in a triangle add up to more than 180, two-angles pop up, etc.

In physics, grand narratives can be found among those telling about the effect of forces, e.g. gravity, producing parabola orbits on earth, and circular and ellipse orbits in space. Jumping from a swing is a simple example of a complicated model. The grand narratives of physics enabled the rise of the Enlightenment period, and of the modern democratic society replacing religion with science.

In economics, an example of a grand narrative is Malthus' 'principle of population' comparing the linear growth of food production with the exponential growth of the population; and the Keynes model relating demand and employment creating the modern welfare society. As are the macroeconomic models predicting the effects of different taxation and reallocation policies.

Also limit-to-growth models constitute grand narratives predicting the global economical and ecological future depending on different production, consumption, and pollution options.

### **Comparing Modern LIB-mathematics and Enlightening LAB-mathematics**

Set-based LIB-mathematics has as number-sets integers, rational numbers etc. Multiplicity-based LAB-mathematics only has stack-numbers and per-numbers. In LIB-mathematics the order of introducing the operations are +, -, \*, /. In LAB-mathematics the order is the opposite. In LIB-mathematics two digit numbers are introduced before mathematics. In LAB-mathematics it is opposite. LIB-mathematics considers multi-digit numbers as being easy and algorithms as being difficult. In LAB-mathematics it is opposite. LIB-mathematics introduces fractions and decimals and percentages late. LAB-mathematics introduces decimals and fractions in grade 1 as parallel ways of counting leftovers when bundling. LIB-mathematics adds fractions. LAB-mathematics adds per-numbers. LIB-mathematics solves equations by neutralizing. LAB-mathematics solves equations by backward calculations. LIB-mathematics talks about multiplicative structures and proportionality. LAB-mathematics just recounts. LIB-mathematics postpones calculus to late in secondary school. LAB-mathematics introduces 1digit calculus in primary school. LIB-mathematics postpones trigonometry to late in secondary school. LAB-mathematics introduces trigonometry in primary school. LIB-mathematics believes that sets, fractions and functions are fundamental

mathematical concepts that are applied in many areas. LAB-mathematics is set-free, fraction-free and function-free. In LIB-mathematics  $2+3$  IS 5 and  $2*3$  IS 6. In LAB-mathematics  $2+3$  can be whatever depending on the units, while  $2*3$  IS 6 since the unit here is 3s:  $2*3 = 2 \text{ 3s} = 6 \text{ 1s} = 6$ .

### Learning Principles: Grip & Grasp and Gossip

The learning principles are ‘grip&grasp’, and ‘learn from gossip’. In primary school children as mammal offspring learn by doing. This means that learning has to come through the hands, ‘greifen vor begreifen’, both as objects you can grip and as actions you can perform. Thus a LAB-approach means that the written learning material in primary school is brief since the learning takes place, not by reading, but gripping and moving. And in secondary school texts have the form of gossip.

One example of grip&grasp is the progression from 1.order over 2.order to 3.order counting. 1.order counting means counting in 1s rearranging sticks to form an icon. Thus the five-icon 5 contains five sticks if written in a less sloppy way. In this way icons are created for the numbers until ten that becomes a very special and strange number having its own name, but not its own icon.

2.order counting is counting by bundling&stacking using icon-bundles. First we bundle e.g. 7 sticks in e.g. 3-bundles, in 3s. Then we stack the total in two stacks: a stack of 3s, and a stack of unbundled singles. The stacks may then be placed in a left bundle-cup and in a right single-cup.

In the bundle-cup a bundle is traded, first to a thick stick representing a bundle glued together, then to a normal stick representing the bundle by being placed in the left bundle-cup.

Now the cup-contents is described by icons, first using cup-writing 2)1), then using decimal-writing to separate the left bundle-cup from the right single-cup, and including the unit:  $T = 2.1 \text{ 3s}$ .

->           ->		->          ) -> ■■  ) ->  )  )
Or with icons:		-> 2 3s + 1 1s -> 2x3 + 1x1 -> 2)1) -> 2.1 3s

Later also bundles are bundled, calling for a new cup to the left. Thus 4 5s can be rebundled in 6 3s and 2 1s, i.e. as 6)2), where the 6 3-bundles can be rebundled into two 3-bundles of 3-bundles, i.e. as 2))2 or 2)0)2), leading to the decimal number 20.2 3s: ||| ||| |) -> |) | |).

Including ten as bundle-size means going on from 2.order counting, using bundles with both a name and an icon, to 3.order counting, using the bundle-size ten having a name, but not an icon. Before introducing ten as 10, i.e. as the standard bundle-size, 5 is chosen as the standard bundle-size together with a sloppy way of writing numbers hiding both the decimal point and the unit so that e.g. 3.2 5s becomes first 3.2 and then 32, thus introducing place values where the left 3 means 5-bundles and the right 2 means unbundled singles. This leads to the observation that the chosen bundle-size does not need an icon since it is never used when using place values: 1, 2, 3, 4, bundle.

In secondary school young people learn by listening, but the sentences need to be gossip with known subjects, so they tell you something new about something you already knew. This means that abstract concepts must be presented as abstractions from examples and not as examples of abstractions, i.e. from below and not from above.

### **Learning Steps**

A student's learning-process has five steps: Do, name, write, reflect and communicate. On top of that a teacher should be able to design a learning experiment for the students, and be able to learn from observing it being carried out. The experiment is performed three times, first by the designing teacher, then by a student communicating with the teacher, finally by two students communicating with each other while the teacher is observing. During all three experiments the teacher looks for examples of cognition, both existing recognition and new cognition.

Afterwards the student teacher works out a learning report reporting the three experiments observed. The report finally formulates a hypothesis based on what has been learned from observing these three experiments. For each of the 2\*4 final CATS-reports this hypothesis is validated by arranging a new learning experiment to be tested on one student and on two students; and by comparing the prediction from the hypothesis with the observations. Example:

Do: take 5 sticks from a matchbox and arrange them, first next to each other, then as the symbol 5. Say: five sticks can be rearranged to the number symbol or the number icon 5. Write:  $T = 5$ .

Reflect. That five sticks are called five is old cognition. That five sticks can be rearranged as the number symbol 5 is new cognition. That the number-symbols are icons containing the number of sticks they describe is new cognition. Also it is new cognition that this makes a fundamental difference between the ability of numbers and letters to represent the world: Numbers are icons representing what they describe; words are sounds often installing what they describe.

Communicate. Write a postcard: 'Dear Paul. I have just participated in an experiment where I was asked to take out five sticks from a matchbox and arrange them as the number symbol 5. All of a sudden I realized the difference between the symbols '5' and 'five', the first representing what it describes and the second representing a sound. See you next week. Best wishes'.

Design an experiment. 'Build the first twelve number-symbols by rearranging sticks'.

Hypothesis. This experiment will help Peter, who has problems understanding 2digit numbers. Once he tries to build a number symbol for ten, eleven and twelve, he will realize how smart it is to stop inventing new symbols and instead begin to double-count bundles and unbundled.

Test. After having finished reporting what Peter did and said, it is my impression that constructing the number icon for ten was what broke the ice for Peter. It seems as if this enabled him to separate number-names from number-icons, since it made him later ask, 'Why don't we say

one-ten-seven instead of seventeen? It would make things much easier.’ This resonates with what Piaget writes: ‘Intellectual adaptation is thus a process of achieving a state of balance between the assimilation of experience into the deductive structures and the accommodation of those structures to the data of experience (Piaget 1970: 153-154)’.

## **PYRAMIDeDUCATION**

In PYRAMIDeDUCATION, 8 student teachers are organized in 2 teams of 4 students choosing 3 pairs and 2 instructors by turn. The coach helps the instructors instructing the rest of their team. Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both recognition and new cognition, i.e. both assimilation and accommodation. The coach helps the instructors when correcting the count&add problems. In each pair each student corrects the other student’s routine-assignment. Each pair is the opponent on the essay of another pair. Having finished the course, each student teacher will ‘pay’ by coaching a new group of 8 student teachers.

## **Basic Teacher Training or Master Degree**

The CATS-approach in teacher training can be used during the basic teacher training; or it can be used as a master degree for persons having already finished their basic training and wanting to become a math-coach offering part of the CATS approach as in-service training to other teachers. The training material will be available, partly at the MATHeCADEMY.net website, partly at a university licensed to offer the CATS-approach as part of a franchising agreement.

## **Conclusion**

What is called ‘mathematics education’ is instead ‘metamathematics education’, trying to fulfill the Platonic dream of presenting the world as examples of mathematics, itself being examples of the metaphysical form set; but instead making students turn their backs to the subject that through its number-predicting ability is the foundation of the modern industrialized well-fare society. To become what it says, mathematics education should respect the nature of mathematics as a natural science grounded in its physical root multiplicity, as was the case in the Enlightenment period.

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