DIFFERENTIAL EQUATIONS AS A TOOL FOR MATHEMATICAL MODELING IN PHYSICS AND MATHEMATICS COURSES

A study of high school textbooks and the modeling processes of senior high school students.

Ruth Rodríguez Gallegos

University Joseph Fourier, France; Tecnólogico de Monterrey, Mexico

Email: Ruth.Rodriguez@imag.fr

Abstract

This paper proposes a study, based on my doctoral dissertation, which deals with the learning and teaching of mathematical modeling in physics and mathematics courses. It was oriented specifically for the senior high school students in France. In 2002, the new syllabi for the Physics and Mathematics courses emphasized the role of mathematics as a tool for modeling in other sciences. Firstly, a description of the modeling process was established for this work. Secondly, the textbooks commonly used in the Physics and Mathematics courses were analyzed. These analyses revealed the transposition process of the "modeling process" practiced by the experts into a different process adapted for school. The setting up of an experimental situation including some unusual tasks (out of the scope of the common didactic contract) for senior high school students allowed the identification of the influence of the existing "praxeologies" in these classes when students were subjected to problem-solving situations. Some of the difficulties linked to the setting up of this transposition process ware analyzed and are presented in the study.

1. Introduction

Nowadays, society has new expectations about the skills of the young individuals. Particularly, some international studies have established the importance of the development of individual skills to model and solve real life problems (OCDE, 2003).

In 2002, the new syllabi for the Physics and Mathematics courses emphasized the role of mathematics as a tool for modeling in other sciences.

This article proposes a study that deals with the learning and teaching of modeling, specifically in senior high school Math and Physics courses in France.

The aims of this work were to study how the modeling processes "becomes alive" in French schools and to identify the students' difficulties in modeling a real life problem.

2. The framework in use

2.1 Modeling a mathematical modeling

The first step was to establish a description of the "modeling" process in this work. To establish this definition, a review of several research works about the subject was necessary; for instance, Study 14 of the International Commission on Mathematical Instruction (ICMI) and the memories of some international conferences, such as International Congress on

Mathematical Education (ICME) and the International Community of Teachers of Mathematical Modelling and Applications (ICTMA). Finally, this definition was built considering the definitions used by Blum and Niss (1991), Kaiser (1995) and Henry's (2001) works. The final definition to be used in this work is represented in Figure 1 (for a detailed description of this seven-stage process see Rodriguez 2007):

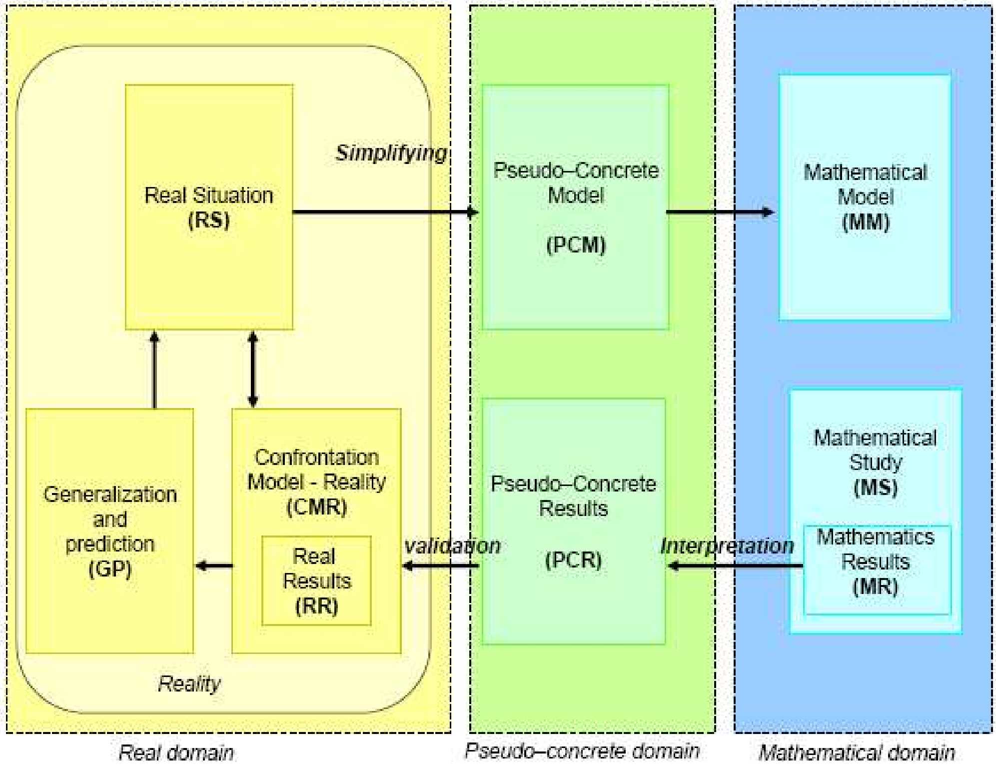

Figure I: Description of the Modeling Process

Based on the description of "mathematical modeling" established before, some of the textbooks commonly used in the Physics and Mathematics courses were analyzed. The results of these analyses allowed the characterization of the proposed modeling process "to be taught" in the senior year of high school. The methodology followed to analyze the textbooks is described in the next section.

2.2 The didactical transposition

This study used the notion of *praxeologies* as a useful tool to analyze textbooks. This notion was taken from Chevallard's anthropological theory (1999) which has been used extensively and further developed by many French researchers such as Artaud (2007). She says that "in this approach, two main aspects are considered. The first one regards what is learnt and taught and is modeled in terms of mathematical praxeologies. The second one is concerned with the learning and teaching activities as such and is modeled in terms of didactic praxeologies" (Artaud, 2007, page 373). A praxeology has four components:

- a number of types of tasks T, which refers to what one has to do
- a technique τ , which provides a way to achieve tasks of the given type

- a technology θ, for every technique, which is the “discourse” that justifies and explains the technique
- a theory Θ, which is the “discourse” that justifies and explains the technology

In order to carry out this analysis and taking into account a first review of the French syllabi and textbooks, the notion of “differential equation” was focused on a modeling tool. This notion is first taught in high school.

The methodology followed in textbooks was identified in a first review. From the analysis of the chapter of “Differential Equations”, it was determined and classified the kind of tasks students are commonly demanded to do. In the second part of the analysis, this classification of tasks was validated by analyzing the content of the chapter. If, in this part, there was some reference to the task, this kind of task was kept since it was representative of a task demanded from students. When the list of tasks was validated, a second detailed analysis took place to identify the techniques utilized for each task, and to, eventually, find out if a technology or theory had been considered in the chapter.

3. The first results: the analysis of textbooks

The analysis of the math textbooks allowed the identification of other types of tasks to be practiced by students in the course:

Type of task	***Transition between phases in the modeling process***	***Description of the task***
T_{DE}	Pseudo-Concrete Model → Mathematical Model	Set up a differential equation which models a real situation in pseudo-concrete terms (found in the explanatory texts of each exercise) that will lead to a Mathematical Model
T_{GS}	Mathematical Model → Mathematical Study	Find a general solution of the differential equation
T_{PS}	Mathematical Model → Mathematical Study	Find a particular solution using an initial condition (given in the explanatory text of the exercise)
T_{AQ}	Pseudo Concrete Domain → Mathematical Model	Answer a question, formulated in pseudo-concrete terms, based on the mathematical results obtained

The task T_{DE} to “set up a differential equation” to model a real situation is rarely demanded from students. Most of the times this equation is given by the exercise; and sometimes this kind of tasks are reduced to “justifying” that the model is an equation given on the statement.

It was also observed that in a Mathematics class, “to solve” a differential equation means to use a theorem previously demonstrated by the teacher in class. Writing a mathematical model (in this case, a differential equation) is an important stage in the modeling process, but this study confirmed the absence of this in-class task, so it was decided to extend the domain of the study to the Physics class. Three textbooks of the Physics class were analyzed, in particular the chapters of “Circuit RC” by using the same methodology followed for the math course. The types of tasks identified in this class are shown in the following table:

Type of task	***Transition between phases in the modeling process***	***Description of the task***
T_{EC}	Pseudo-Concrete Model → Physical Model	Represent an electric circuit scheme diagram (a resistor-capacitor circuit, RC circuit)
T_{DE}	Physical Model → Mathematical Model	Set up a differential equation which models the tension capacitor $U_C(t)$ present in the circuit.
T_{PS}	Mathematical Model → Mathematical Study	Find a particular solution (verifying that a given function in the exercises is the solution of the differential equation
T_I	Pseudo-Concrete Domain ↔ Physical Domain ↔ Mathematical Domain	Determine the intensity of the electric current i(t) in the circuit using the function $U_C(t)$

Therefore, it was evident the need to insert a Physical Domain in the modeling process of reference in order to describe the modeling process that took place in the Physics course.

The original scheme was modified; thus, a “new” description was devised for the next part in this paper, as shown in Figure II.

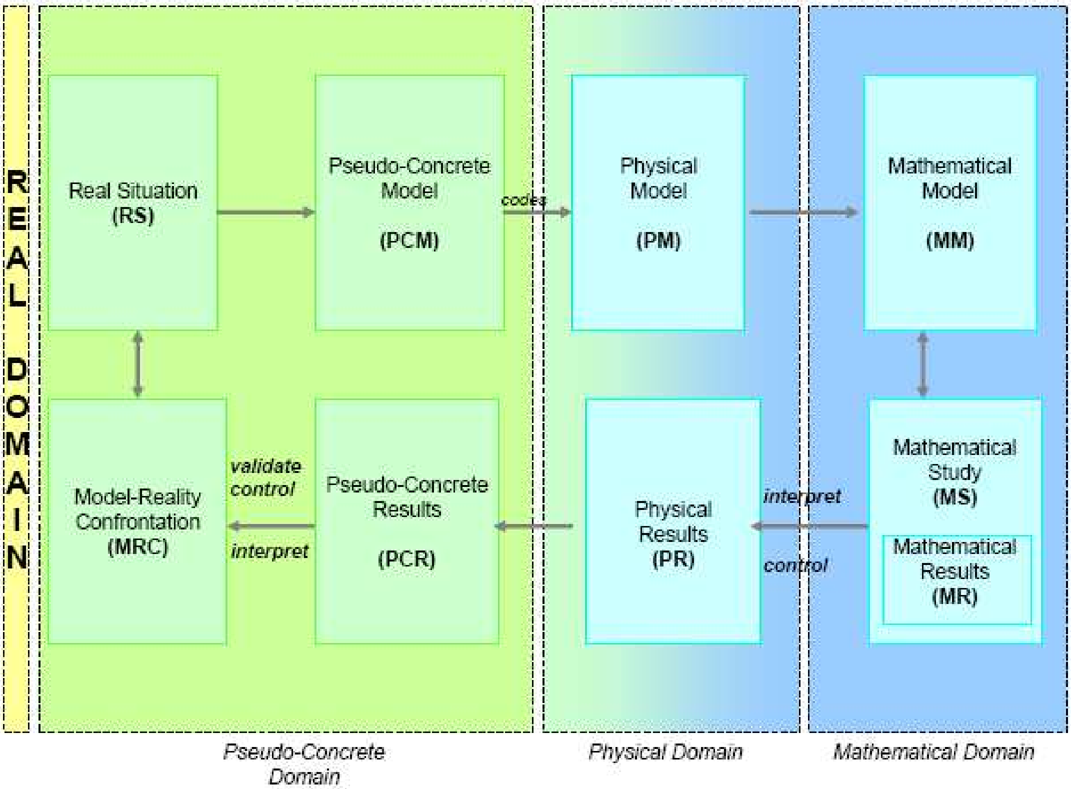

Figure II: Modeling process in a Physics course

It is important to indicate the insertion of the Physical Domain in the original process of modeling. Two stages were included in this domain. The first stage was concerned with the physical model. In the case of the electric circuit, a physical device or a circuit diagram in paper were included. The second stage was related to the mathematical treatment of the differential equation, which modeled the intensity in the circuit. Consequently, some physical results could be obtained. Physics teachers might find this process significant when working with mathematical modeling.

As a conclusion for the part regarding the textbook analyses, these revealed the transposition process (Chevallard, 1991) of the experts' modeling processes into a different process used in schools. Even when the modeling process is practiced more often in the Physics class than in the Math class, some important shortcomings were observed and these are commented below:

- Even if the type of task T_{EC} "Represent an electric circuit scheme" appears in the textbooks, it is a task not commonly assigned.
- The task T_{DE} "Set up a mathematical model" (a differential equation) is usually assigned to the students, but the steps to do it are given in the exercise
- In the Mathematics and Physics classes, there was a lack of exercises to make the student face the transition from the stages of Pseudo-Concrete Results to those of Model-Reality Confrontation. Henry (2001), among other researchers, has considered this transition important from a didactic point of view.

The lack of tasks to make students face modeling situation along with the experts' processes led to the design of an experimental situation, which is described in the next section.

4. A brief description of the experimental situation

The second aim of this work was to identify the students' difficulties to model a real life problem built around the results of the textbooks' analysis. Special attention was paid to the kind of tasks and techniques available in the textbooks and usually presented to students in class.

An experimental situation was set up to include some unusual tasks (out of the scope of the common didactic contract) for the senior high-school students. Following are the three characteristics chosen to design this experimental situation.

1) To confront students with the transition from Real Situation + Pseudo-Concrete Model towards the construction of the Physical Model. This confrontation was normally absent according to the textbook analysis of the physics class.
2) To provide no directions to students regarding the writing of the Mathematical Model (transition from Physical $\rightarrow$ Mathematical Model). A guide was usually observed to establish a differential equation such as a mathematical model of a particular situation in math and physics class.
3) To confront the students to the transition from Pseudo-Concrete Results $\rightarrow$ Model-Reality Confrontation. This confrontation was also absent in both of the analyzed classes.

In the experimental situation, the students were proposed to do the modeling of the functioning of a defibrillator. This electronic device applies an electric shock to restore the rhythm of a fibrillating heart. A description about the functioning of this device was introduced in a text. It explained to the students the mechanism of operation in physical (electric) terms.

After the introductory text, students are given a key question. Then, with a set of five tasks, students are guided to reach the answer for the original question: ***"What is the probability of survival of a man who has a cardiac problem in the street and is assisted with a defibrillator?"***

Out of five tasks, tasks A and B are provided as examples of the following:

Task A

We need to model the defibrillator with an electric circuit, like those studied in class.

Draw an electric circuit diagram and justify your choice.

A possible (and correct) answer for this question was an RC circuit diagram like the one shown in Figure III.

Figure III: Possible answer to question A

It is worthwhile noting the presence of the photo in the figure, showing the use of the defibrillator. The photo might have influenced the students' answers.

Task B

Set up a model (a differential equation) for the tension in the defibrillator. Justify the laws that were used to establish the model.

A possible (and correct) answer for this question was $\frac{dU_C}{dt} + \frac{1}{RC} U_C = 0$.

There were three other verification tasks: if a function given was the solution of the differential equation (task C); about the intensity received by the patient (task D); and finally about the comparison and validation of the results obtained in the activity, that is, the real results given by a source (task E). In the next section, the results for tasks A and B are analyzed (see Rodriguez (2007) for a more detailed analysis of the students' productions).

5. Experimentation settings

The experimentation was done with 25 students in their senior year of three different French high-schools. The setting of the experimentation was done after teaching the topic of "Electrics Circuits". The students worked in pairs and had one hour to solve the problem.

Taking into account the type of tasks usually demanded form the students in the physics class and some elements of the techniques found in the textbooks, the answers to the experimental situation provided by the pairs were analyzed. The analysis also included the final response concerned with had the patient been or not been able to survive. It was of particular interest to determine what difficulties students had found in this kind of tasks and if the textbooks presented some problem-solving technique. Also, it was looked into the ways students solved the problem. Finally, it was interesting to specify what stages (and transitions in between) of the modeling process were more challenging for the senior students.

6. Partial results: the modeling activity of the students

The students' answers for questions A and B allowed the identification of the influence of the existing *praxeologies* in the Mathematics courses, and mainly in the Physics ones, in the students' solving processes.

About Task A

A difficulty observed in question A was that students found it hard to propose a diagram with all the physical elements of the electric circuit such as resistances, capacitors, etc. Some "hybrid" configurations appeared. The difficulty to insert a resistance as part of the electric circuit (replacing the patient's thorax) was recognized from the students' productions. Even if the word "resistance" appeared in the introductory text of the activity, this word did not make specific reference to a physical term. This fact could be observed in the answers provided by two pairs of students:

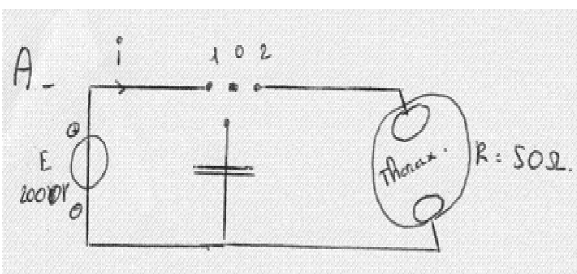

Figure IV

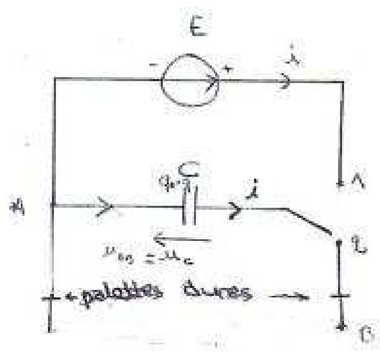

Figure V

In figure IV the thorax is represented by a "circle" and the electrodes are also represented in the diagram. Even if the legend "R = 50 Ω" referred to a resistance, it was difficult for these students to use the "correct" electrical symbol as observed. As well, in figure V, the electrodes were drawn by the students ("*palettes dures*" in French) but in this case, the students did not include any reference to a resistance in their circuit. This pair ignored the place of the patient. Students might not have considered the patient as part of the circuit since the patient had no electrical nature regardless of his importance in this scheme.

Some other students proposed a scheme from a "physical model" such as the one shown in Figure VI:

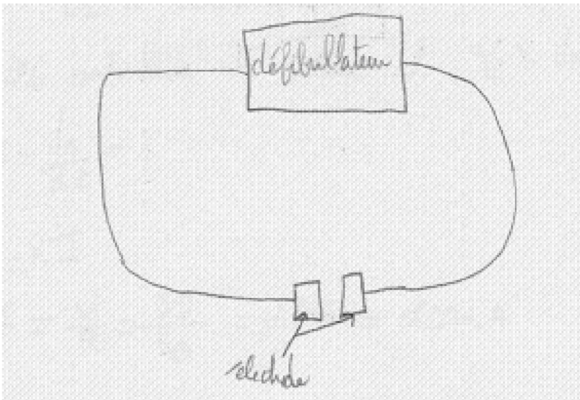

Figure VI

In figure VI, the students were completely in a Pseudo-Concrete Domain. The elements shown in the diagram were not physical elements. This answer could not be considered at all. It made us wonder about the students' comprehension of the task assigned. This difficulty could be located in the transition from Real Situation + Pseudo-Concrete Model (text of the activity) to the writing of a Physical Model. In reference to the observed praxeologies in the analyzed textbooks from the first part, no technique was observed in these manuals to perform this kind of task, even if it was an important step from a modeling point of view.

About Task B

For question B, several difficulties were identified to propose a differential equation to model the capacitor tension in the circuit. Some students forgot the Physique laws to do that as well as how to establish a relation between the magnitudes involved. However, they sometimes used incorrect physical laws or principles. Some pairs of students established a differential equation for the charge capacitor q(t): $\frac{dq}{dt}+\frac{1}{RC}q=0$. It is worthwhile mentioning that this kind of answer in the students' productions was observed since there was no technique in class or in the textbooks suggesting this procedure. It seemed "natural" for the students to establish this differential equation. Once again, no technique was observed in the analyzed textbooks.

Another difficulty observed was the explicit explanation (or its absence) about the selection of the study between the charge or discharge of the capacitor and the resistance. The interest in this case was to study the discharge from the defibrillator to the patient, but an important number of students apparently "forgot" that fact. In many cases, question A worked as the validation for the model proposed and it helped the students "to correct" the differential equation proposed. An example of this is illustrated in Figure VII.

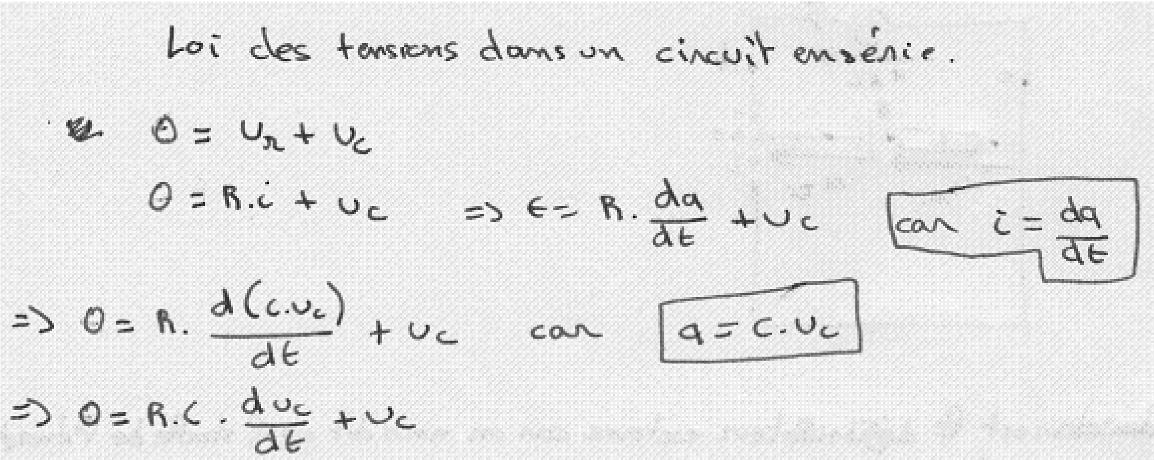

Figure VII

As it can be seen, symbol “E” was erased from the last two equations because these students had written an “E” instead of a “0” (zero) which corresponded to the generator charge from the circuit if the discharge was studied. Task C asked for verification of the solution of the differential equation for a given function. The feedback given for task C allowed the validation or/and correction of the model proposed in task B. This difficulty showed the absence of relationships between the physical phenomena to study (real situation described) and their modeling (discharge of a RC circuit + differential equation).

The students’ solution of this experimental situation showed evidence of the role of the « pseudo-concrete » model for both, the initial real situation and the students’ physical model, which they built by using the modeling approach.

The results can illustrate the importance of having a clear understanding of the situation to be modeled as a consequence, establishing a pseudo-concrete model to build a correct mathematical model. The benefits of the transition between the “objects and world events” (called the real domain in this work) and the “models and world theories” (called the physical and mathematical domains) had already been observed and documented by other researchers like Thiberghien and Vince (2004).

The conclusions of this study agree with these researchers particularly in the importance of students’ self practice of the task “Real Situation -> Physical Model -> Mathematical Model,” which are crucial for the acquisition of the modeling process. However, the exploration of the study revealed that it can be difficult to attain.

6. Conclusions

To conclude, as described, this study agrees with Chevallard’s work (1991) about the type of modeling that is finally taught (“taught” knowledge) in the Physics and Mathematics courses. It also stands for an important gap in regards to the experts’ modeling processes (the “wise” knowledge). The process of modeling defined as a point of departure in this work can be modified according to some issues observed in the student’s activities.

It is also important to emphasize how to define or where to find definitions for the experts’ practice, topic which is commonly debated amongst the mathematical modeling community members. The definition can be customized depending on the scientific field where a specific situation needs to be modeled. This must be an interesting subject to discuss and develop in future modeling works.

Another important issue in this work is the importance of both, the construction of an adequate pseudo-concrete model and the physics model which leads students to model it themselves. This study finds that it is absolutely necessary that the aforementioned be done in school. Secondly, giving feedback from one task onto another develops a proper solution of each task. This is an important feature to take into account in the design of future modeling activities. In future papers, the relevance of teachers' external interventions to help students overcome their difficulties could also be discussed.

Learning acquisition is learning through practice. Therefore this study recommends the design of activities with all possible modeling phases as well as teacher training courses in this topic.

References

Artaud, M. (2007). Some conditions for modeling to exist in mathematics classrooms. *Modeling and Applications in Mathematics Education. The 14th ICMI Study* (371-378). New York: International Commission on Mathematical Instruction ICMI.

Blum, W. & Niss, M. (1991). Applied mathematical problem solving, modeling, applications, and links to other subjects – State, trends and issues in mathematics instruction. *Educational Studies in Mathematics 22(1)*, 37-68.

Chevallard, Y. (1991). *La transposition didactique - Du savoir savant au savoir enseigné*, deuxième édition. Grenoble : La Pensée Sauvage éditions.

Chevallard, Y. (1999). L'analyse de pratiques d'enseignantes en théorie anthropologique du didactique. *Recherche en Didactique des Mathématiques 19(2)*, 221-266.

Henry, M. (2001). Notion de modèle et modélisation dans l'enseignement. In Henry, M. (Ed.), *Autour de la modélisation en probabilités* (149-159). Besançon : Commission Inter-IREM Statistique et Probabilités.

Kaiser, G. (2005). *Mathematical Modeling in school – Examples and Experiences*. (Article issued at a seminary of the Education Department, University of Hamburg, Hamburg, Germany).

Organisation for Economic Co-operation and Development [OCDE] (2003). *The PISA 2003 Assessment Framework-Mathematics, Reading, Science and Problem Solving Knowledge and Skills*, 194. Retrieved November 30, 2007, from http://www.oecd.org/dataoecd/46/14/33694881.pdf

Rodriguez, R. (2007). *Les équations différentielles comme outil de modélisation en Classe de Physique et des Mathématiques au lycée : une étude de manuels et de processus de modélisation en Terminale S*. Doctoral dissertation, University Joseph Fourier, Grenoble, France.

Tiberghien, A., & Vince, J. (2004). Études de l'activité des élèves de lycée en situation d'enseignement de la physique. In Pugibet, V. et Gettliffe-Grant, N. (Eds.), *Cahiers du Français Contemporain (*10). ENS Editions.