ICME-11

Paper for Topic Study Group 21: Mathematical applications and modelling in the teaching and learning of mathematics

TEACHING TO REINFORCE THE BONDS BETWEEN MODELLING AND REFLECTING

Mette Andresen, head of NAVIMAT, University College Copenhagen, Denmark, mea@cvukbh.dk

Abstract: When students engage in mathematical modelling in problem solving settings, they tend to focus on the application aspects and on the results of the modelling process. The recent introduction in Denmark of compulsory use of CAS in upper secondary school mathematics may amplify this risk, so that students develop a bare technical and practical view on mathematical models and modelling, at the expense of a more profound understanding of the nature of mathematical activity and thinking. This paper advocates a teaching of models and modelling, which balances out the 'technical-application's view on mathematics by explicit reflections upon the use of models and upon the modelling process. As a basis for such balanced teaching, a new model is suggested in the paper. The basic idea of it is to combine levels of mathematical activities, well known from Realistic Mathematics Education, with levels of reflections, presented in a recent work on philosophy of mathematics (Prediger 2007). Further, the design of a teaching sequence, dealing with a specific mathematical model, is outlined in the paper. Teaching experiments based on the model are planned, but still not carried out, in a small-scale qualitative research project by the author.

BACKGROUND FOR THE STUDY

Growing interest in models and modelling in Danish upper secondary

Danish upper secondary school was subject to a reform in 2006. One element of the reform was the introduction of multi-disciplinary projects, which also involved mathematics. Mathematical models and modelling were explicitly mentioned as part of the curriculum in mathematics, although there were no formal requests for specific modelling sequences or themes; the teachers autonomously decide about the planning at that level of details. Consequently, the scene was set for the teaching of models and modelling in close connection with other subjects like physics, chemistry, social sciences, economy etc.: teachers who wants to, and who feels competent to, can carry out teaching sequences on modelling and/or on the investigation of authentic mathematical models. To some degree, this was also the case before the reform. The new point is that since all teachers are now obliged to carry out multi-disciplinary projects, a growing number of teachers, apparently, turn their interest to modelling issues (Andresen and Lindenskov 2007) .

A bare technical view on modelling as a potential drawback of the use of CAS

The 2006-reform also implied the introduction of compulsory use of computer algebra systems (CAS) in mathematics. This introduction will, obviously, cause comprehensive changes now and in the future. For the teaching of models and modelling, the use of CAS opens up for a wider range of topics, and for numerical treatment of a variety of models, like for example in the case of differential equations models. It has potentials for a huge extension and development of the teaching of models and technical modelling in the sense of comparing a number of models and fitting them with a set of data (Andresen 2007 p5). It also has potentials to support students' model recognition and capability to understand and criticize authentic use of ready-made models in different contexts.

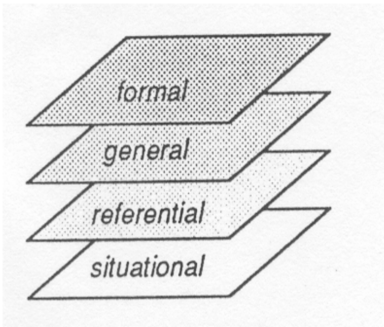
Results from our previous research, though, show that in general, the use of CAS tends to change focus of attention into technical and practical aspects of upper secondary school

mathematics. This tendency results from the individual teachers' choices based on preferences, habits and CAS competencies. In general, teaching with computer is centred upon solving tasks, whereas the reading of proofs and theoretical treatments in general are carried out without use of computer (Andresen 2006 p 28). Thus, there is a potential danger, that the same trend might direct the teaching of 'models and modelling' into a bare 'application' view on mathematics by the students, at the expense of giving the students a more profound insight into mathematical activities, theory and knowledge.
To avoid this, the students' more technical and practical view on models and modelling, partly caused by the introduction of CAS, can and should be balanced out by explicit reflections upon the use of models and upon the modelling process. In the following, Gravemeijer's four level model of mathematical activity (Fig.1) is combined with a four level stratification model of mathematical reflections, discussed in (Prediger 2007). Combination of the two models serves as a basis for a teaching model that aims at such balance.

FOUNDATIONS

The role of reflections in learning mathematics

The role of reflections in learning mathematics is apparent in the domain-specific instruction theory for realistic mathematics education (RME). RME is rooted in Hans Freudenthal's idea of '*mathematics as a human activity*'. According to this theory, students should be given the opportunity to reinvent mathematics by mathematizing; mathematizing subject matter from reality (horizontal mathematizing) and mathematizing mathematical matter (vertical mathematizing). This implies that the students develop a high level of intellectual autonomy. Hence, the core principle is that mathematics can and should be learned on one's own authority, through one's own mental activities.



(Fig 1.) Levels of activity. Gravemeijer, K. & Stephan, M. (2002). p 159

Horizontal and vertical mathematizing may be modelled by the passing of four levels of activity (fig.1). A new mathematical reality is created at each level.
Reflections substantiate the progressive mathematizing (Gravemeijer 2002 p 147 ff).

Stratification of mathematical reflections combined with the four levels of activity

The use of philosophical reflections as a tool for mathematical reasoning was recently discussed (Prediger 2007). Prediger's discussion was based on the stratification in (Neubrand 2000) of reflective practice in mathematics into four levels:

1) The level of the mathematician
2) The level of the deliberately working mathematician
3) The level of the philosopher of mathematics
4) The level of the epistemologist.

The teaching model, which was presented in (Andresen and Froelund 2008), involves the preparation of a reflection guide. Basic to the preparation of the guide is the combination of Neubrand's four levels of reflection with the four levels of activity in Gravemeijer's model, as it is illustrated in

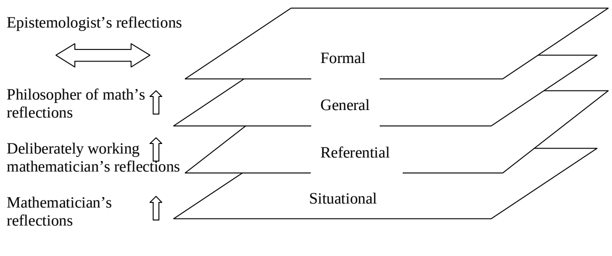


Fig 2 The combined model

(Fig 2.): the reflections should initiate or support students' activities to pass from one level to the next in Gravemeijer's model. Thus, reflections at level 1 in Neubrand's model bring the student from the situational to the referential level. Next, reflections at level 2 bring the student from the referential to the general level. Finally, reflections at level 3 bring the student from the general to the formal level. It should be noticed, that (Andresen and Froelund 2008) does not suggest a fifth level on top of the 'formal' level in our combined model. Rather, the epistemological reflections are considered to widen the formal level in its horizontal dimensions. This attests the authors' view, that epistemological reflections do not and should not represent an external or additional level to adequate mathematical reflections (see also Prediger 2007 p 45).

REALISATION

Use of the combined model

Use of the combined model is not restricted to teaching sequences on modelling in the 'applications' sense of the concept. According to RME, mathematics is a human activity and since the vertical and horizontal mathematising are main heuristics for learning mathematics, all mathematical activities imply some sorts of modelling. The combined model, therefore, may be used in all teaching sequences. To illustrate this point, the following example of using it takes a rather traditional series of tasks as its starting point. We will demonstrate how it may be used to stress the modelling aspects by explicit reflections upon traditional mathematical activities.
The basic idea for the teacher, during the design of the students' learning trajectory, is to pick out moments of interest for making the reflections explicit. The term 'moments of interest', here, relates to the levels in Gravemeijer's four-level model or, more precisely, to situations with special potentials for the students to rise or descend from one level to another. Stimulation of reflections relevant to the level, and the efforts of making them explicit, will not only support the students' concept formation and learning, but also aim to enhance the students' awareness, knowledge and consciousness about mathematical activity as such.
Since the students' reflections should be regarded as intellectual activities, carried out autonomously by the single individual, we intend to stimulate the reflections by making the teacher ask thought-provoking questions. The questions form a reflection-guide, following our combined model. When designing the teaching sequence, the teacher prepares the reflection guide.

Preparation of a reflection guide

A thorough analysis of the teaching materials in case must precede formulation of questions for the reflection guide. The analysis aims to identify potential levels of students' mathematical activity as they are illustrated in (Fig1.). We take as our starting point that the reflection guide should be tailored to fit the teaching materials, not vice versa. Hence, the task to identify potential activities may in some cases become an issue of interpretation, to discern the textbook's rationale. Or it may start with an imagination of the students' hypothetical learning trajectory. In the following, we give an example of preparation of a guide based on a calculus worksheet with tasks, picked out from (Christiansen et al. 2006), translated from Danish in Encl1.

Identification of potential activity and formulation of thought-provoking questions

Thought-provoking questions at all four levels of reflection are based on the potential activity levels. In the following, four groups of thought-provoking questions, one group for each of the levels in Prediger/Neubrands model, are formulated in relation to the levels of activity, identified and picked out from Encl1's text.

The tasks in Encl1 appear like traditional word problems and concern with optimisation. So, some of the questions, especially at the two first levels, may appear traditional, plain and naïve. In this demonstration, we consider working with the whole worksheet as being one, overarching activity. According to its authors, the worksheet was meant for students' training (Crhistiansen et al. 2006, the website). A realistic estimation of the duration of the students' work with accomplishing the worksheet would be two lessons and the rest as homework – depending, of course, of the group of students.
What we find interesting here is that the questions to the students can give rise to reflections upon the worksheet's tasks as a whole, as well as reflections upon the single tasks, elements of solving them etc. Therefore, the scene is set for reflections at all four levels in Prediger/Neubrand's model, even if the worksheet deals with short, traditional word problems and exercises rather than a full modelling cycle.
The students' knowledge about the levels of reflections, their consciousness about their own reflections and the thinking they initiate, are key points for the outcome.

Questions at the level of the mathematician.
To deepen the students' understanding of the rise from a situational to a referential model, questions at first level in Prediger's model should be asked. In terms of RME, this rise is horizontal mathematising, where the model emerges.
Examples of *activities* at the situational level in Encl1:

- In task 1; talking about the total of two numbers, the cube of each of the numbers and the total of the cubes of the numbers
- In task 2; talking about the boat situated 10 km from the coast, the house 12 km along the coast, the distances along the beach, across the water and the total distance he has to go, the fastest movement and the time it takes.
- In task 3; talking about a certain point of the parabola and its distance to another point.
- In task 6; talking about a box and its length, width, surface and about how much it contains

In this context, *questions* that stimulate reflections at the level of the mathematician, which means stimulate rising to the referential level, could be like these:

- Concerning task 1: what could we call the two numbers? How could the cube of the first number be denoted? The cube of the second? How can we express that the total of the two numbers is 12?
- Concerning task two: how could we denote the distance he has to go, crossing the water? How can we express the time this part of the tour it takes?

To deepen the students' understanding of the mathematising, *questions* might be asked that stimulate descending from referential level to situated, like for example:

- Concerning task 4: What is the shape and orientation of the parabola? How is the parabola situated relative to the point?
- Concerning task 5: what does A mean? What would happen in line 21 if *h* equals zero?

Questions at the level of the deliberately working mathematician.
To identify activities at the referential level, generated by the worksheet, the teacher has to build on his or her imagined learning trajectory. In our example, a plain interpretation of the intended trajectory involves traditional activities at this level. Examples of such *activities* are:

- In task 1; putting up the equations $a + b = 12; s = a^3 + b^3$ where a and b are still interpreted as the two, unknown numbers and s is thought of as the total of the numbers' cubes

- In task 2; making a drawing like Fig3 and put up expressions like

$$y = \sqrt{x^2 + 100}; t_1 = \frac{\sqrt{x^2+100}}{4}; t_2 = \frac{12-x}{8}; t_1 + t_2 = \frac{12 - x + 2\sqrt{x^2+100}}{8}$$

Where, for instance, t_1 is still thought of as the time spent on crossing the water and t_{1+} t_2 is interpreted by the students as the total time spent on the tour.

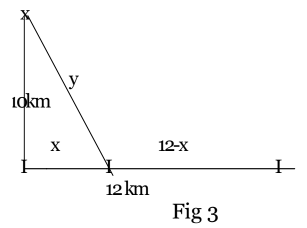


Fig 3

- In task 5; finding the derivative of A, substituting 500 for V and isolating r^2in the equation in line 23

In this context, *questions* that stimulate reflections at the level of the deliberate mathematician, which means stimulate rising to the general level, could be like these:

- Concerning task 1: Which one is the variable, if we see *s* as a function? That is, what does *s* depend on?
- Concerning task 2: How can we determine the derivative of t_{1+} t_2?
- Concerning task 7: How could *f(v)* be maximised? What are the restrictions on the variable *v*, if any?

Again, *questions* might also be asked, which stimulate descending from general to referential level, and maybe descending further to the situational level, like for example:

- Concerning task 2: Can t_1 be negative? What would the maximum value be for x?
- Concerning task 3: How many solutions can there possibly be?
- Concerning task 5: What are the restrictions on *r* and *h*?

Discussions of how to discern between the three levels may support the individual student's deliberate changes between all three levels, and prepare for the next level of reflections.

Questions at the level of the philosopher of mathematics

Rise from general to formal model tends to happen over time, sometimes in a somehow subtle way. By solving the tasks in Encl1, certain routines are carried out repeatedly. The overarching activity to accomplish the worksheet's tasks, therefore, gives insight into corresponding methods. Reflections upon the emergence of a method from carrying out a routine a number of times, give insight into mathematics at formal level. So, questions that stimulate reflections upon methodological issues such as how to identify a method and how to distinguish between different methods support the students' rise from activity at the general level to activity at the formal level.

In our case, *questions* at the level of the philosopher of mathematics, therefore, could be like:

- What do you need to know, to be able to minimise an unknown quantity?
- What kind of expressions can be maximised?
- How do you state the assumptions and put up the conditions for optimisation?
- Is it always possible to put up an expression and find the derivative? What exceptions can you imagine?
- Do you know any analogous to this way of problem solving?

Questions at the level of the epistemologist

Activities at the formal level may be widened by further reflections. The characteristics of mathematics and related issues can be enlightened by classroom discussions of questions like:

- Do you know any examples of modelling from other subjects than mathematics?
- What are the aims of models in other subjects like physics, chemistry, economy, and text analysis, social sciences?
- Who decide about the validity of a model in each of these subjects? What are the 'rules'?
- How can you decide whether a model belongs to mathematics? What are the characteristics?

- Who make the models, what are they made for?

Design of teaching with the reflection guide

The detailed design has to be carried out by the teacher, since the teaching must be tailored to fit the actual group of students in its unique situation and context. One of the main questions the teacher has to decide about with regard to the structure of the sequence is: *when* are the students supposed to become aware of the reflection process - under the process or after it is accomplished? The teacher's choice must be conditional on the complexity of the sequence, the level of difficulties with the content and the expected outcome for the students. Different designs of the teaching sequences give weight to different levels of reflections. Hence, the weighting of reflections is another important question that regards the learning goals. For example, reflections at the first two levels may serve to develop technical aspects of the 'mathematical modelling competency' whereas reflections at the other two levels may serve to throw light on modelling processes. In general, reflections at the higher levels tend to be more general than the others since the very notion of reflect means '*transcending the immediate object of the present consciousness in a learning subject. The outcome of a reflection is a consciousness at a more general level. At this level the object at the first level is situated in a broader context. Consequently knowledge of the external relations of the first object to similar objects in the same class is now produced.*' (Andresen and Froelund 2008 p3).

In our example, the teacher has to decide when the students are supposed to answer the guide's questions, and how to organise this part of the teaching sequence. One way to do it could be to arrange a classroom discussion, introduced by the teachers' presentation of an adapted version of the basic idea. Another way could be to let the students write an essay, if the formal regulations include that sort of activity in mathematics. The preparation of a reflection guide based on our model helps the teacher to clarify the learning goals and to be aware of and realise hidden potentials for the students' learning. In line with the above discussion of the need for a more balanced teaching, the model can serve to draw students' attention to what might be called core mathematical activities. Nevertheless, there are no obvious reasons to assume that the model would not work in other parts of the modelling processes too.

The questions in the first group, supposedly, resemble 'normal' questions, asked by any math teacher. None of the other questions in the reflection guide have to be a stroke of genius, neither. The point is for the teacher to make the students aware of their own thinking, and to encourage them to develop new insight during discussion in the classroom.

CONCLUSION

Naturally, use of the combined model to prepare a reflection guide, should and will be followed by a try out in the classroom. The first try out of the model is planned to take place in spring 2008, in connection with a research project with focus on the teaching of authentic models in upper secondary. The actual case concentrates on probability and estimation in relation to statistical models. Another try out is planned in the same period, in an experimental, multi-disciplinary project involving mathematics and philosophy. The philosophy class will concentrate on mathematical knowledge, reflections and philosophy, whereas the mathematics class will base some of the lessons on guided reflections in accordance with the combined model. These two settings seem obvious for reinforced bonds between modelling and reflecting. The above example, though, demonstrates that it is possible to accentuate the modelling aspect of mathematical activities even in a prearranged 'exercise-setting'. In this case, the reflection guide's questions deliberately stress the student's experience of underlying ideas, basic principles and generalisation. The guided reflections in this case intend to enhance and focus those experiences, which the students in all cases might gain at random, when making series of exercises.

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Calculus 4

Task 1:
The total of two numbers is 12. Determine the two numbers under the assumption that the total of their cubes is smallest possible.

Task 2:
A man sits in a boat 10 km from the coast. He wants to reach a house, situated 12 km along the coast, compared to his actual position. How far along the beach should he land, assumed that he can walk 8 km/h and row 4 km/h, to arrive the fastest possible, and how long time does it take?

Task 3:
Find that point of the parabola $f(x) = 2x^2$, which is closest to the point (2; -1)

Task 4:
Find that point of the parabola $f(x) = -x^2 + 2x$, which is closest to the point (3; -8)

Task 5:
The area of a cone's surface may be determined by:

$$A = \pi \cdot r \cdot \sqrt{r^2 + h^2}$$

and the cone's volume by

$$V = \frac{1}{3} \cdot \pi \cdot r^2 \cdot h$$

Determine radius and height of a cone with the smallest possible surface, given that the volume is 500.

Task 6:
A box, containing 2 L, without lid must have a length, double of the width. Determine the width so that the area of the box's surface is smallest possible.

Task 7:
The number of cars passing a hindrance is given by:

$$f(v) = \frac{500v}{0,007v^2 + 0,2v + 5}$$

where *v* is the speed in km/h. Determine the speed which allows the largest number of cars to pass the hindrance.

Results:

1: 6 and 6	2: 5,77 km and 3,66 hour	3: (0,338; 0,228)	4: (3,97; -7,84)
5: 6,96 and 9,86	6: 11,4 cm	7: 26,7 km/h	