PROGRAPH DIAGRAMS – A NEW OLD SYSTEM FOR TEACHING FUNCTIONAL MODELING

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Abstract:

This paper shows the basic concept of Functional Modeling in mathematics education which gets more and more important. Hence of this significance it is necessary to think about adequate graphical methods to explain the fundamental idea of a function and its influence to values and other functions. PROGRAPH diagrams are a very good possibility to explain these combinations effectively. The basic idea of a function, composition of functions or returning values of a function can be shown impressively. It is also possible to find easy nested diagrams for recursive functions. So the mathematical sight to topics which can be handled through functions gets opened and the possibilities for working with functions get more voluminous. With the help of given examples the idea of this type of diagram becomes clearer and through implementation with CAS the effective use of functions is demonstrated. Hence the combination of PROGRAPH diagrams and CAS allows an understanding insight to functions so that the Black-Box-Principle is eliminated. For a modern and effective education Functional Modeling is absolutely necessary. With the combination of graphical representation and CAS learners are able to see and learn the common foundation of mathematical ideas and applications.

1. Functional Modeling

One of the basic ideas of functional modeling is the idea of functional thinking which has been published by Hans-Joachim Vollrath [11] but can also be found as projection a little bit earlier in the dissertation of Karl Josef Fuchs [2]. If functional modeling is done with students it will be necessary that they will be able to practice functional thinking. For this reason anyone could explain "Functional Thinking" as the pattern for mental activity in the process of "Functional Modeling".
Following the definition of functional thinking given by Vollrath we have to focus on a:

- Methodological aspect,
    - Dependency of parameters,
    - Idea of systematic – dynamical variation,
- Phenomenological aspect,
- Quantitive aspect,
- Input-Output aspect.

And as K. Fuchs writes in his paper 'Functional Thinking – a fundamental idea in teaching Computer Algebra Systems' it is necessary to add one more aspect for a better understanding [3]:

- The Algorithmically Aspect

Another fundamental idea which has to be followed is the idea of modeling. There exists a lot of literature for this idea like my dissertation [9] or my paper of Pecs [10]. So I want to characterize it shortly. Models in mathematics can be seen as additives for the array, appliance and advancement of theories. Models provide a basis for demonstration and description of contents.
Through the combination of these two fundamental ideas it is possible to create the idea of functional modeling which is a deeply and pivotal idea in Mathematics and Computer Science

[4]. Thereby it is obvious to work with CAS where the order of execution of functions and composed functions can be seen as a program. Models should be facile, clear and intuitional on one hand meaningful on the other hand. Hence we have to split the construction of a functional model in two phases:

- **Black-Box-Phase**
  We look at the function as a whole. The formal description is done through a consequent use of data-flow-diagrams, the so called PROGRAPH diagrams [7] but the function stands for a "machine" where values are shoveled in from one side and tumbling out from the other side. The process which is going on inside this machine is of no interest in this phase. This can be found in the Input-Output aspect of Vollrath.
- **White-Box-Phase**
  It is profitable for exploring the inner structure of a model. Through working and acting automatically the interactions and references of the components of a model can be explored.

These both principles have been described by B. Buchberger [1] in conjunction with CAS for the first time.

2. What are PROGRAPH diagrams?

For the process of implementation it is absolutely necessary to use diagrams as mentioned above. Every function can be described in a data-flow diagram like:

- What is the aim of the function?
- How is it implemented?

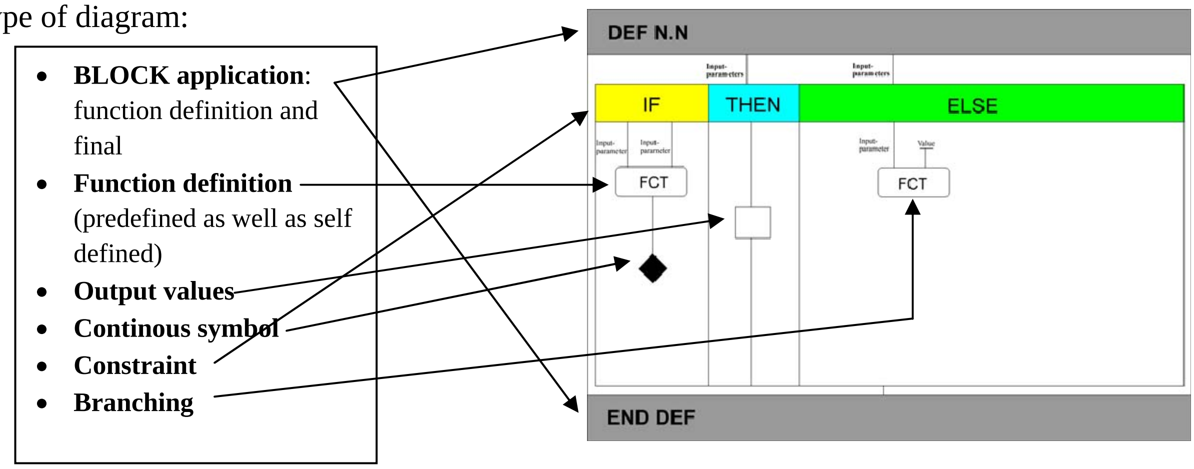
A very suitable way to see these aspects evidently is PROGRAPH. Arguments for using this type of diagrams are:

- The input parameters process continuously through the application of functions. Matwin [5] calls it 'principle of single assignment'. Fuchs [3] writes in this case: 'You can haunt a once determined value staying unchanged through the rest of the functional system.'
- Each function returns only one value. You may say it meets the 'definition of a function in Mathematics'.
- Recursion can be done easily because - as Fuchs writes - 'You can experience the 'Picture within a Picture' – structure as a visual metaphor for recursion easily`.

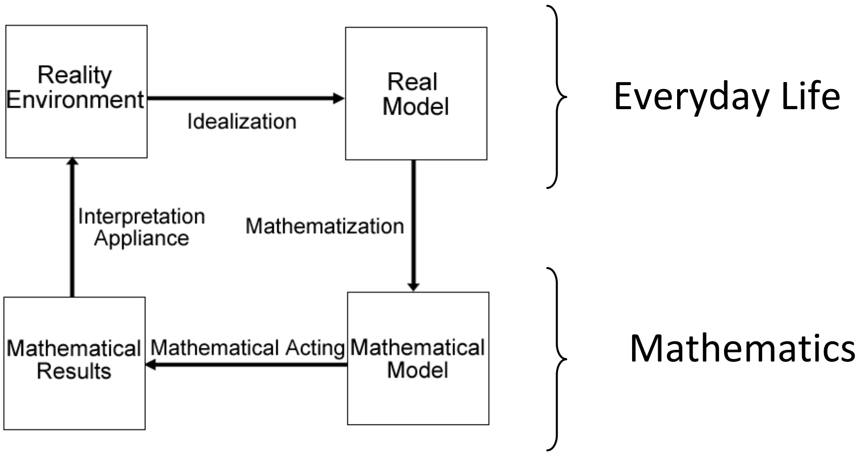
Another point I want to add is that you can recognize the function declaration easily:

Definiendum	Definition Mark	Definiens

All the PROGRAPH diagrams have the following style. In the given picture you can see a special PROGRAPH diagram which I have designed to show the prototypical elements of this type of diagram:



I don't discuss this diagram type at this point because for a mathematician it is self-explaining and through the following examples the structure will become clear at all.
Let us have a look at the scheme of a modeling process:



With the help of such PROGRAPH-diagrams it is possible to simplify the step of "Mathematization". So students get an additive for this step. They have to find the variables and the functional parts in the "Real Model" so that they can formulate and prepare a "Mathematical Model". The structuring through abstraction with the help of flow-chart-diagrams so that proper functions can be found is a very helpful way for students.

3. Examples for Working with PROGRAPH diagrams and CAS

The treatment of examples of probability is often a stepchild in computer aided mathematics education. When they are encouraged through the usage of a computer the programs for experimenting are already prepared and the pupils don't have to think about the mathematical backgrounds. They use such programs as Black-Boxes and the White-Box sight on topics in probability gets lost completely. For a better understanding of activities in probability it's necessary to discuss certain topics like Laplace-contribution or coin-flipping intensively.

a. Flipping a Coin

If the learners are well experienced in these fundamental themes they can solve the following example.

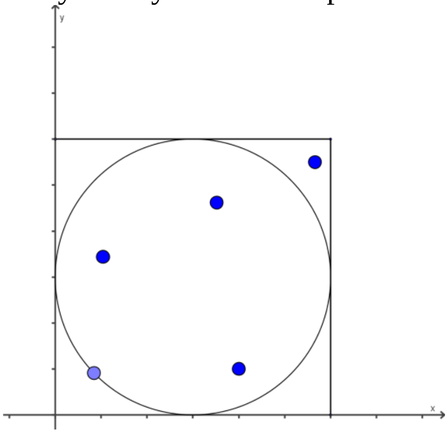
A small field of a board has the shape of a circle with radius R. A coin is thrown on this field. Think about a program which is able to determine whether the coin is located within or on the boarder (outside) of the inner circle field.

The first two steps of the process of modeling and the part of functional thinking in this process should be shown with the help of this example. Therefore I take the graphic from above and show the steps of the process explicitly:

1st Step: Reality/Environment:
In the first step the students have to understand the situation given in the example. It is obvious to recognize that there is no given specific description for a situation in reality. But this is not very problematic because through an easy modification the example could look like the following:

A small dart field has the shape of a circle with radius R. An (dart) arrow is thrown on this field. Think about an implementation that helps you to analyze whether the coin is located within or on the boarder (outside) of the inner circle field.

For realizing the solution of this problem the students have to think about the opportunities of such a coin. Maybe they will draw a picture like the following:



After thinking about this fact they can go to the second step with the help of simplifying the facts. In case of the dart field it would mean that they do not think about "special inner fields" of the dart field.

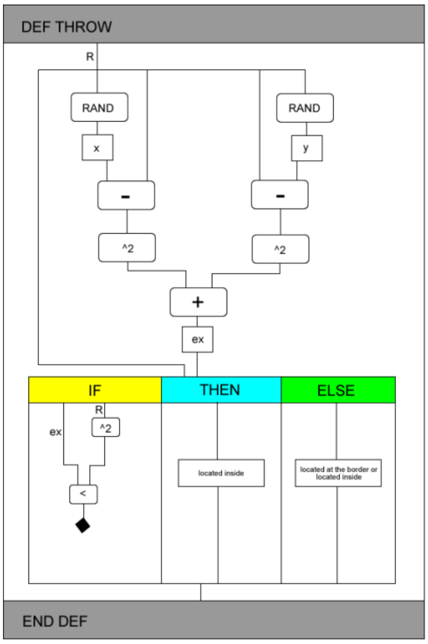
2nd Step: Real Model

The realization of such an implementation is in the first instance a brainteaser. So the students have to think about the location of the coin at the field and its description through coordinates. Depending to the input-parameter it is able to modify the example in the way to find the coin outside the field. For the first consideration this arrangement is too difficult. So the learners should think about the case whether the coin can be located inside or at the border of the board.

With the help of PROGRAPH diagrams the mathematical model is created in the next steps respectively the students switch with the help of these diagrams from the "Real Model" to the "Mathematical Model". This part is shown in the third step.

3rd Step: Mathematical Model

Hence a circle board was chosen for this example it is possible to get the equation for the circle board as $(x - R)^2 + (y - R)^2 = R^2$ if the circle is positioned like shown in the graphic.

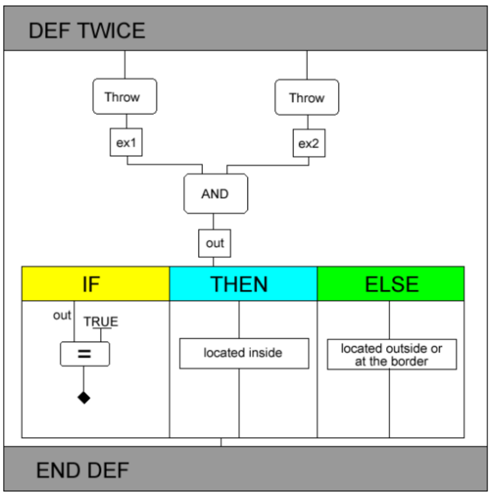


The left part of this equation is substituted with the expression ‘ex’. For the output it is obvious that we get Boolean values, because if the coin is located in the board, TRUE is the common output otherwise FALSE is enough. Such an example could be described very easy through a PROGRAPH-diagram in the first step. This leads to an understanding White-Box-sight on such procedures. The implementation is specific to the used program or CAS. So we have another argument to draw a PROGRAPH-diagram, because of the general sight to such functional topics. The diagram could look like the one drawn on the right margin:

As said before the output for this module is a Boolean data type {TRUE, FALSE}. After realizing this part it is possible to expand this example because it is not necessary to have a look at only one coin. An expansion of this example would be the view on two coins which are thrown to the circle board. The possibilities we are able to scan for are:

- 2 coins located inside the field,
- 1 coin located inside, 1 coin located outside the field,
- 2 coins located outside the field.

If we are interested in the coins located inside the circle board the case differentiation will get easier because if TRUE is unreturned in at least one of the cases it will be sure that not less than one coin is located outside the circle field. The PROGRAPH diagram is a following diagram of the one yet given and looks like the following:



With the help of these diagrams the implementation is possible in every CAS, because the structure of these examples doesn't change.
So this type of diagram is also well adapted for the third step, because the functional structure and the functions are becoming obvious for students. In the 4^{th} step they only have to interpret their solutions.

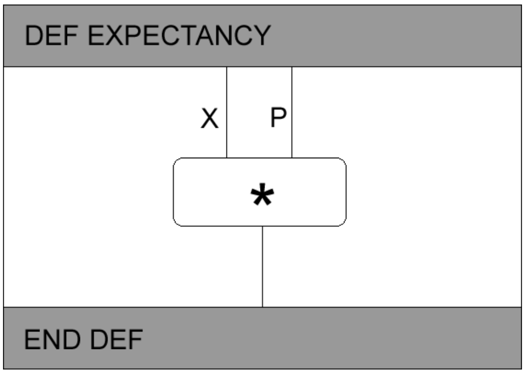
Another very important point which can be seen evidently in this example is the fact of modularization [8]. Modularization means dividing a problem into little parts which can be handled separately. Here I have chosen the other way. We have created a second part out of a main part. These two composed parts can be fit together and if you look at these parts as a joint problem it will be easy to see the main problem of throwing two coins. This modular construction system could be implemented in lot problems given in Mathematics or Informatics. Especially difficult problems in mathematical education can be handled more easily. Functions also can be handled easier if an informatical view is allowed to them in mathematical education. It is obvious to see that the functions in this view have an explicit module character.

b. Expectancy and Variance

Another topic of probability is the topic of distributions. There certain parameters have to be calculated very often because they are of immanent importance for the interpretation of these distributions. Such parameter values are expectancy and variance.
Expectancy and variance of well known distribution (binomial distribution, standardized normal distribution) can be calculated very easy through the input of the known formulas. But if an empirical distribution is given, it will be necessary to assess the values of expectancy and variance with the help of the given definitions for $p_1 = P(X=x_1)$, $p_2 = P(X=x_2)\dots p_n = P(X=x_n)$ and $X = \{x_1, x_2\dots x_n\}$:

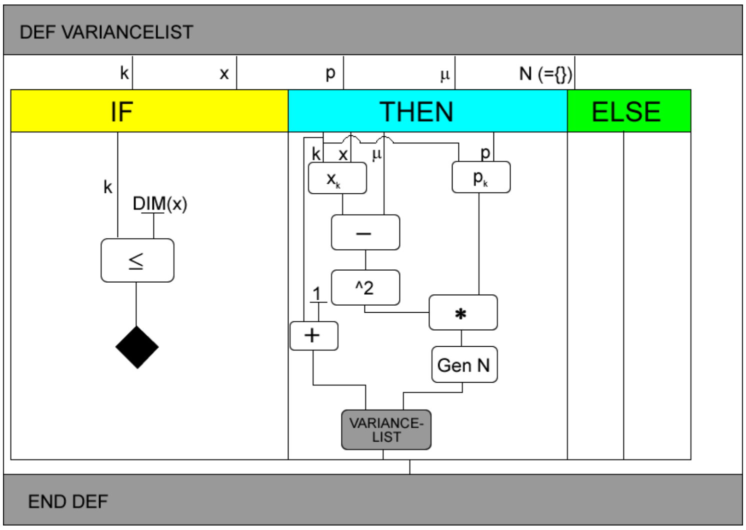
- $\mu = E(X) = x_1p_1 + x_2p_2 + \cdots + x_np_n$,
- $V(X) = (x_1-\mu)^2p_1 + (x_2-\mu)^2p_2 + \cdots + (x_n-\mu)^2p_n$.

When E(X) is specified it is obvious that this is an inner product of the two lists $\{x_1, x_2\dots x_n\}$ and $\{p_1, p_2\dots p_n\}$. If we understand the two given lists as two vectors and define it like vectors with a CAS it is to calculate the solution. For the variance it is necessary to formulate a Sum-function like the following:
$V(X) = Sum((x_k-\mu)^2p_k, k, 1, Dimension(X))$
If this fundament is created it is possible to have a look at the PROGRAPH diagrams. First we look at the expectancy:

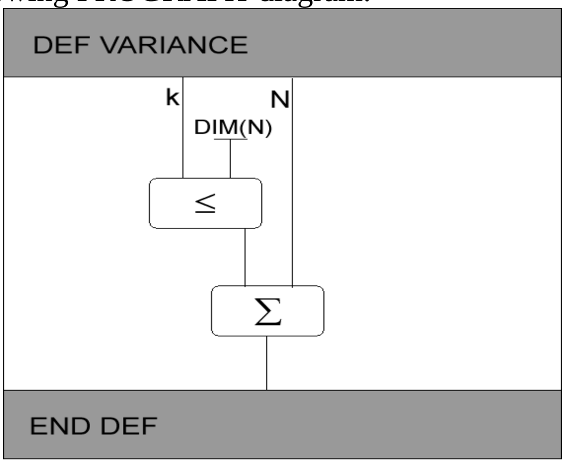


The PROGRAPH diagram for the implementation of the variance is because of the structure of the formula a little bit complicated. It is necessary to split it into two parts.

The first part creates a list of n elements $\{(x_1-\mu)^2 p_1, (x_2-\mu)^2 p_2, \ldots, (x_n-\mu)^2 p_n\}$ out of an empty set N. This list is filled consecutively through the function **Gen N**. The filled list can be called N again. The function $\mathbf{x_k}$ and $\mathbf{p_k}$ extracts the element in position k of the two input-lists x = $\{x_1, x_2, \ldots, x_n\}$ and p = $\{p_1, p_2, \ldots, p_n\}$. After composing this list in the second part it is possible to sum up the elements of list N. The PROGRAPH diagram for the first part could look like the following:



The second part of adding the elements is the easy part of calculating the variance. It is represented by the following PROGRAPH-diagram:



After realizing these diagrams it is very easy to implement these functions in any CAS used in education. Only the specific notation in the different programs has to be observed. But if the learners are common with one system it won't be a problem to implement the functions.

4. Epilogue

Through the division of the modeling process into a Black-Box- and a White-Box-Phase it is possible to get a clear, intuitive and on the first sight understanding description of this process which is meaningful and detailed.
In the first part, the Black-Box-Phase, the question `Which information is for which component?' and 'Which information is necessary for another part of the process?' can be answered. The inner structure of the used components is not answered yet. This is an exercise

for the second part, the White-Box-Phase, where the functions are implemented as a source-code. The special use of PROGRAPH diagrams is very efficient in the first part because the structure can be seen on the first sight. The functional character of the involved parts appears because it is easy to see that each function is fed through a unique assignment of input parameters. The advantages for using PROGRAPH diagrams are:

- Easy to learn because there are less basic concepts. Especially there are no allocations, no loops or no skips.
- Higher efficiency because the source code which should be implemented is very short compared to an imperative program code.
- Higher trustiness because considerations or proofs of the correctness of the source code is easier because of the mathematical background.

Like Nassi-Shneiderman [6] diagrams for imperative modeling PROGRAPH diagrams return a very bright picture of the processes which should be described. The structure of the drawn diagrams can be implemented 1:1 in a CAS. It doesn't depend on the system.

Another very important point which has to be articulated is that students or pupils are highly motivated and challenged by Functional Modeling also problems can be very difficult in description, shown in 3.b. in implementation.

Through the combining of diagram construction and Computer Algebra Systems the teaching with the help of computers or graphical calculators gets more exciting again. For students it is more efficient because mathematical facts will be understood through informatical handling. The interdisciplinary aspect of Mathematics and Informatics gets starched and new forms of education are possible.

Another important point of Functional Modeling is delivered by the implementation of Standards in mathematical education which is done right now in Austria and has been done in other countries like Germany.

If we have a closer look at those we can find a lot of interesting points at the website of the NCTM (www.nctm.org) it is obvious to see the listed points which K. Fuchs has mentioned in his paper [3]:

- Students should learn an ambitious common foundation of mathematical ideas and applications,
- Students need to understand the mathematical concepts of the function,
- Students should be adept visualizing, describing and analyzing situations in mathematical terms.

Through the attendance of Functional Modeling I am sure that these Standards can be achieved and more understanding for mathematical circumstances can be created.

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