TEACHING MATHEMATICS IN THE CLASSROOM THROUGH PROBLEM SOLVING

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ABSTRACT

We present an approach to teaching-learning-evaluating mathematics through problem solving. The historical context is briefly described leading up to the current guidelines of the *National Council of Teachers of Mathematics* (USA), on which we base a characterization of teaching **through** problem solving. Considered a teaching method, its foundations and general guidelines for implementation in the classroom are presented. Mathematics teaching-learning-evaluation through problem solving has been used and studied systematically, at all educational levels and in teacher education activities, by the Problem Solving Work and Study Group (*Grupo de Trabalho e Estudos em Resolução de Problemas* – GTERP), based at UNESP, Rio Claro, São Paulo, Brazil. The research developed by the group follows essentially qualitative approaches, with the main objective of reflecting on and analyzing the possibilities this method offers for increasing learning and improving teaching processes, as well as promoting improvement of the practices of mathematics teachers. Our studies show that students' construction of knowledge related to mathematical concepts and contents is more meaningful and effective, and when applied in teacher education activities, it favors significant improvements in their teaching practice.

INTRODUCTION

Mathematical problems have occupied a central position in school mathematics curricula since Antiquity. Records of mathematical problems can be found in ancient Egyptian, Chinese, and Greek history, and problems are still found in mathematics textbooks from the 19th and 20th Centuries, and up to the current day. According to Stanic and Kilpatrick (1989), the main point to consider in the examples presented in these books is they reflect a very limited view of learning and problem solving. The role of problem solving in the school mathematics curriculum is the result of conflicting forces linked to old and enduring ideas regarding the benefits of studying mathematics and a variety of events that occurred at the beginning of the 20th Century.

Problem solving as a classroom teaching method, here denoted Mathematics Teaching-Learning-Evaluation through Problem Solving, is a fairly new concept in mathematics education, despite the long history of problem solving in school mathematics. Consequently, the method has not been the object of very much research.

PROBLEM SOLVING AND RESEARCH IN THE 20TH CENTURY

The definition presented by Leder (1998) for "educational research" outlines the various aspects that it can address, including: the purposes of education, teaching and learning processes, professional development, organizational resources, policies, and strategies. The volume of studies on library shelves indicates that it has become an immense enterprise, and that the search for new knowledge not only continues, but has been widely documented. In recent years, the volume, scope, and diversity of educational research, in general, and research in mathematics education, in particular, have grown substantially.

Having begun as a field of systematic studies with the work of Felix Klein at the beginning of the 20th Century, mathematics education had grown into a vast and intricate endeavor by the end of the century. Felix Klein was one of the most important mathematicians of the late 19th Century, and one of the last, together with Gauss, Riemann, and Poincaré, to break the barrier of specialization and provide the fundamental elements that gave impetus to the mathematics of the 19th and early 20th Century. It was then that he wrote his book *Elementary Mathematics from an Advanced Standpoint.*

Klein stated, in his autobiography written in 1923, that the totality of all knowledge and the ideal of a complete education should not be neglected because of specialized studies, and that universities should be concerned with preparatory teaching in the schools. He emphasized teacher education, in particular. He was a brilliant mathematician who was sincerely and seriously concerned about issues related to teaching.

At that time, mathematics teaching was characterized by work based on repetition, where the memorization of facts was considered important. Years later, a different orientation began to emphasize that students should learn with comprehension, and should understand what they were doing.

It was then that talk began of solving problems. George Polya (1944) emerged as a reference, emphasizing the importance of discovery and of encouraging students to think by means of problem solving. In his book *How to Solve It*, he states "A great discovery solves a great problem, but there is always a bit of discovery in the solution of any problem". In 1949, he wrote that solving problems is the specific realization of intelligence, and that if education does not contribute to the development of intelligence, it is obviously incomplete. In 1948, the work developed by Herbert F. Spitzer in basic arithmetic, in the U.S.A., was based on learning with comprehension, always using problems-situations; and in 1964, in Brazil, the teacher Luis Alberto S. Brasil defended teaching mathematics using problems that generated new concepts and contents.

In the 1960s and 1970s, mathematics teaching in Brazil and in other countries of the world was influenced by a reform movement known as Modern Mathematics. This reform dominated the scene, and like the others before it, failed to include the participation of classroom teachers. It presented a mathematics based on structures of logic, algebra, topology, and order, and emphasized set theory. It highlighted many properties, reflected excessive concern with mathematical abstractions, and used a universal, precise, and concise language. However, it emphasized the teaching of symbols and complex terminology, which compromised learning. In this reform, teaching was approached with excessive formalization, distancing itself from practical issues.

According to Onuchic & Allevato (2005), these reforms were not as successful as expected. The questions continued: Are these reforms aimed at preparing a citizen who is useful to the society in which he/she lives? Do they seek to teach mathematics in a way that prepares students for a world of work that demands mathematical knowledge? In addition to this, particularly in the 1970s, there was a growing concern with a mathematics curriculum that was initially aimed at increasing test scores, also known as computational ability tests.

Concomitantly, at the beginning of the 1970s, systematic investigation of problem solving and its implications for curricula was initiated. Thus, the importance attributed to problem solving is relatively recent, and only in this decade did mathematics educators come to accept the idea that the development of problem-solving abilities deserved more attention. At the end of the 1970s, problem solving emerged, gaining greater acceptance around the world. In 1976, at the *3rd International Congress on Mathematical Education*, in Karlsruhe, Germany, problem solving was one of the themes addressed.

Discussions in the field of mathematics education in Brazil and around the world demonstrated the need to adapt school work to new trends that could lead to improved ways of teaching and learning mathematics.

In the U.S.A., the National Council of Teachers of Mathematics (NCTM) responded to this concern with a series of recommendations for the progress of school mathematics, published in 1980 in a document entitled *An Agenda for Action* (NCTM, 1980). All interested groups and individuals were called to collaborate in the work to seek, through a massive cooperative effort, a better mathematical education for all. The first recommendation in the document was that "*problem solving be the focus of school mathematics in the 1980s*."

During the 1980s, many resources were developed to facilitate work with problem solving in the classroom, such as collections of problems, lists of strategies, suggestions for activities, and guidelines to evaluate student work involving problem solving. Much of this material contributed to helping teachers make problem solving the central point of their work.

Nevertheless, possibly due to differences in conceptions regarding the significance of problem solving becoming "*the focus of school mathematics*", the work during that decade failed to achieve a good level of progress (ONUCHIC, 1999). Schroeder & Lester (1989) presented three different approaches to problem solving that help us reflect on these differences: theorizing about problem solving; teaching how to solve problems; and teaching mathematics through problem solving. Teachers who teach about problem solving seek to emphasize Polya's model, or a derivation of it. When teaching how to solve problems, the teacher concentrates on the manner in which mathematics is taught, and what of this can be applied in the resolution of routine and non-routine problems. In this view, the essential purpose for learning mathematics was to be able to use it. As the 1980s ended with all these recommendations for action, researchers began to question the teaching and the effect of strategies and models, and to discuss the didactic-pedagogical perspectives of problem solving.

Problem solving, as a teaching method, became a focus of research and studies in the 1990s. This new view of mathematics teaching-learning was based especially on studies developed by the NCTM that culminated in the publication of the *Standards 2000*, officially *Principles and Standards for School Mathematics*.(NCTM, 2000). Problem solving was emphasized as one of the standards for the process of teaching mathematics, and teaching through problem solving was strongly recommended. (ONUCHIC; ALLEVATO, 2005).

Drawing on the ideas of the NCTM *Standards,* the PCN (National Curriculum Parameters) were created in Brazil (BRASIL; 1997, 1998, 1999), which pointed to the development of the capacity to solve problems, explore them, generalize from them, and even propose new problems based on them, as one of the purposes of mathematics teaching. They indicated problem solving as the point of departure for mathematics activities, and discussed ways to do mathematics in the classroom.

Today, at the beginning of the 21st Century, some of the greatest challenges faced by mathematics educators in past decades have persisted, changed, or proliferated, as teaching and society have grown more complex. In "*Unfinished Business: Challenges for Mathematics Educators in the Next Decades*", Kilpatrick & Silver (2000) outline what they believe to be the main challenges: guarantee mathematics for all; promote students' understanding; maintain balance in the curriculum; use evaluation as an opportunity for learning; and develop professional practice.

Cai (2003) emphasizes, however, that although little is known regarding how students attribute meaning and learn mathematics through problem solving, many ideas associated with this approach – the change in the teacher's role, the selection and elaboration of problems, collaborative learning, among others – have been researched intensively, offering answers to various frequently-asked questions regarding this way of teaching.

MATHEMATICS TEACHING-LEARNING-EVALUATION THROUGH PROBLEM SOLVING

The compound word "teaching-learning-evaluation" was chosen intentionally to express the idea that teaching and learning should take place simultaneously during the construction of knowledge, with the teacher as guide and the students as co-builders of this knowledge. In addition, this methodology integrates a more updated conception regarding evaluation. It is constructed during the problem solving, integrated with the teaching to follow students' growth, increasing learning and reorienting practice in the classroom when necessary.

According to Van de Walle (2001), truly effective mathematics teachers should involve four basic components in their work: (1) an appreciation of the discipline of mathematics itself, meaning "doing mathematics"; (2) an understanding of how students learn and construct ideas; (3) the ability to plan and select tasks so that students learn mathematics in a problem solving environment; (4) the ability to integrate evaluation with the process to increase learning and improve teaching from day to day.

Teaching-Learning-Evaluation of Mathematics through Problem Solving differs from approaches that privilege rules regarding "how to". It "reflects a tendency to react to past characteristics, like a set of facts, mastering algorithmic procedures, or the acquisition of knowledge through routine or mental exercise". (ONUCHIC, 1999, p.203).

It corresponds to work in which a problem is the point of departure for learning, and the construction of knowledge occurs in the process of solving it. Teacher and students develop the work together, and learning takes place collaboratively in the classroom (ALLEVATO, ONUCHIC, 2007; ONUCHIC; ALLEVATO, 2005). The methodology is similar to the Japanese approach to teaching mathematics through problem solving. In *Problem Solving as a Vehicle for Teaching Mathematics: a Japanese Perspective*, Yoshinori Shimizu (2003) writes that "Japanese teachers in elementary schools often organize an entire mathematics lesson around multiple solutions to a single problem in a whole-class instructional mode. This organization is particularly useful when introducing a new concept or a new procedure during the initial phase of a teaching unit."(p.206).

For Van de Walle (2001), a problem is any task or activity for which students have no prescribed or memorized methods or rules, and no perception that a specific method for arriving at the correct solution exists. Adding a subjective character to this question, in the context of the methodology presented here, we consider that problem refers to "everything that we do not know how to do, but are interested in doing".

There are no rigidly defined ways to put this methodology into practice (SHIMIZU, 2003; KRULIK; RUDNICK, 2005; ONUCHIC; ALLEVATO, 2005; VAN DE WALLE; LOVIN, 2006). One proposal is to organize activities according to the following stages:

1) Form groups and hand out the activity. The teacher presents the problem to the students, who, divided into small groups, read and try to interpret and understand the problem. It should be emphasized that the mathematical content necessary, or most appropriate, to solve the problem has not yet been presented in class. The problem proposed to the students, which we call the generative problem, is what will lead to the content that the teacher plans to construct in that lesson.

2) Observe and encourage. The teacher no longer has the role of transmitter of knowledge. While students attempt to solve the problem, the teacher observes, analyzes students' behavior, and stimulates collaborative work. The teacher mediates in the sense of guiding students to think, giving them time to think, and encouraging the exchange of ideas among students.

3) Help with secondary problems. The teacher encourages students to use their previous knowledge, or techniques that they already know, to solve the problem, and stimulates them to choose different methods based on the resources they have available. Nevertheless, it is necessary to assist students with their difficulties, intervening, questioning, and following their explorations, and helping them to solve secondary problems when necessary. These refer to doubts presented by the students in the context of the vocabulary present in the statement of problem; in the context of reading and interpretation; as well as those that might arise during the problem solving, e.g. notation, the passage from vernacular to mathematical language, related concepts, and operational techniques, to enable the continuation of the work.

4) Record solutions on the blackboard. Representatives of the groups are invited to record solutions on the blackboard. Correct as well as incorrect solutions, as well as those done for different processes, should be presented for all the students to analyze and discuss.

5) Plenary session. The teacher invites all students to discuss solutions with their colleagues, to defend their points of view and clarify doubts. The teacher acts as a guide and mediator in the discussions, encouraging the active and effective participation of all students, as this is the richest moment for learning.

6) Seek consensus. After addressing doubts and analyzing resolutions and solutions obtained for the problem, the teacher attempts to arrive at a consensus with the whole class regarding the correct result.

7) Formalize the content. At this moment, called "formalization", the teacher makes a formal presentation of the new concepts and contents constructed, highlighting the different operative techniques and properties appropriate for the subject.

It should be reiterated that, in this methodology, the problem is proposed to the students before the mathematical contents necessary or most appropriate for solving it (planned by the teacher according to the program for that grade level) have been formally presented. Thus, the teaching-learning of a mathematical topic begins with a problem that expresses key aspects of this topic, and mathematical techniques should be developed in the search for reasonable answers to the problem given.

For Van de Walle (2001), problem solving should be seen as a main teaching strategy, and he points to the importance of beginning the work from the point where students are, contrary to other ways of teaching that begin from the point where the teachers are, ignoring what the students bring with them to the classroom. He goes on to state that teaching with problems has great value, and that despite the difficulties, there are good reasons to engage in the effort.

Without a doubt, teaching mathematics through problem solving is an approach consistent with the recommendations of the NCTM (NCTM, 2000) and the Brazilian National Curriculum Parameters (BRASIL, 1997, 1998, 1999), as the mathematical concepts and abilities are learned in the context of problem solving. bThe development of thinking processes at the university level will be promoted through these experiences, and the work of teaching mathematics will take place in an environment of investigation guided by problem solving.

In agreement with Krulik & Rudnick (2005), and always with the objective of carrying out ongoing evaluation, new problems related to the generative problem are proposed to the students following the stage of formalization, with the aim of analyzing whether or not the essential elements of the mathematical contents introduced in that lesson were understood. In our view, the understanding of mathematics by students involves the idea that understanding is essential relating. It should be emphasized that indications that a student understands, misunderstands, or fails to understand specific mathematical ideas emerge often as he/she solves a problem.

Rather than being the focus of mathematics teaching, when considered as a teaching method, problem solving makes understanding its central focus and objective. In this way, the emphasis attributed to problem solving is not removed, but its role in the curriculum is broadened. It moves beyond a limited activity to engage students in the application of knowledge, following the acquisition of certain concepts and techniques, to be a means of acquiring new knowledge as well as a process in which the students can apply previously constructed knowledge (ONUCHIC, 1999).

Mathematics has always played an important role in society. This role is more significant today, and may become even more important in the future. People do not always think mathematically, nor are they aware that, if they were to do so, they might make better decisions. This lack of awareness may be a failure of the mathematics taught as well as the way it is taught. Often the teaching of mathematics produces students with over-simplified conceptions and strategies that are excessively mechanical to resolve problems. For Hiebert & Behr(1989), rather than considering knowledge as a package that is ready and finished, teaching should encourage students to construct their own knowledge.

RESEARCH AND SCIENTIFIC PRODUCTION RELATED TO PROBLEM SOLVING AT UNESP

Romberg (1998) considers the objective of mathematics education to be the production of new knowledge about the teaching and learning of mathematics, and that since students learn most of “their mathematics” in school, research should identify the main classroom components that promote mathematical understanding, and point out the organizational characteristics that impede or contribute to the good functioning of those classrooms.

In this paper, we intend to highlight Mathematics Teaching-Learning-Evaluation through Problem Solving, a teaching method which has been used and studied systematically, at all educational levels and in teacher education activities, by the Problem Solving Work and Study Group (*Grupo de Trabalho e Estudos em Resolução de Problemas* – GTERP), based at UNESP, Rio Claro, São Paulo, Brazil. The group has been the generative nucleus of investigations, scientific production, and continuing education in problem solving.

The research conducted composes part of a larger, broader project whose main objective is to reflect on and analyze the possibilities offered by Mathematics Teaching-Learning-Evaluation through Problem Solving for increasing learning and improving teaching processes, as well as promoting improved practices among teachers in the mathematics classroom.

These studies, using a qualitative methodology, consist essentially of interventions in the realm of participant research and action research. Problem solving activities are elaborated by the researchers and applied in the classroom. Thus, a large part of the theses, dissertations, and other work produced by the group narrate and analyze situations involving pedagogical interventions realized by its members in the classroom or in the sphere of teacher education. The scientific production encompasses a wide variety of mathematical contents at all educational levels – elementary, high school, and higher learning.

This set of research constitutes a wide spectrum of research possibilities in mathematics education. A description of the theses and dissertations, as well as the work carried out by the group up until 2005, can be found in Onuchic (1999) and Onuchic & Allevato (2005).

CONCLUSIONS

Considering mathematics educators are people who are professionally concerned about mathematics teaching and learning at any educational level, we can testify to their dedication and relevant production by the volume and quantity of studies in mathematics education carried out in the 20th Century. Students are currently the beneficiaries of the large variety of instructional materials created. Certainly if we compare teachers at the beginning of the century with the teachers of today, we can say that they are much better prepared pedagogically and mathematically. The majority of school mathematics curricula are richer than those at the beginning of the century. In spite of all this, the same complaints are heard today: that students do not like and do not learn mathematics well enough; that teachers do not know mathematics and do not know how to teach it; that school curricula are superficial, repetitive, and fragmented… All these complaints, and data obtained from other sources (research, evaluations, etc), suggest that students leave school poorly prepared, not knowing how to make use of the mathematics they worked with throughout their many years of schooling. As we already said, people are often unable to make decisions in life. These people do not always think mathematically, nor are they aware that, if they did, they would make better decisions.

The teaching method presented here constitutes a way of working in the classroom using generative problems as a point of departure. Using Mathematics Teaching-Learning-Evaluation through Problem Solving, students' construction of knowledge related to mathematical concepts and contents occurs more meaningfully and effectively. Experiences in research with students and teacher education activities, in which this approach was used, have favored significant advances in the understanding of mathematical concepts and contents and improvements in teaching practice.

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