FUTURE DIRECTIONS AND PERSPECTIVES FOR PROBLEM SOLVING RESEARCH AND CURRICULUM DEVELOPMENT

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ABSTRACT

Since the 1960s, numerous studies on problem solving have revealed the complexity of the domain and the difficulty in translating research findings into practice. The literature suggests that the impact of problem solving research on the mathematics curriculum has been limited. Furthermore, our accumulation of knowledge on the teaching of problem solving is lagging. In this first discussion paper we initially present a sketch of 50 years of research on mathematical problem solving. We then consider some factors that have held back problem solving research over the past decades and offer some directions for how we might advance the field. We stress the urgent need to take into account the nature of problem solving in various arenas of today's world and to accordingly modernize our perspectives on the teaching and learning of problem solving and of mathematical content through problem solving. Substantive theory development is also long overdue—we show how new perspectives on the development of problem solving expertise can contribute to theory development in guiding the design of worthwhile learning activities. In particular, we explore a models and modeling perspective as an alternative to existing views on problem solving.

INTRODUCTION

Research on mathematical problem solving has received a good deal of attention in past decades. Among the notable developments have been Polya's (1945) seminal work on how to solve problems, studies on expert problem solvers (e.g., Anderson, Boyle, & Reiser, 1985), research on teaching problem solving strategies, and heuristics and fostering metacognitive processes (e.g., Charles & Silver, 1988; Lester, Garofalo, & Kroll, 1989), and, more recently, studies on mathematical modeling (e.g., Lesh, in press; English, 2007). Existing, long-standing perspectives on problem solving have treated it as an isolated topic, where problem solving abilities are assumed to develop through initial learning of concepts and procedures followed by practice on "story problems," then through exposure to a range of strategies (e.g., "draw a diagram," "guess and check"), and finally, through experiences in applying these competencies to solving "novel" or "non-routine problems." As we discuss later, when taught in this way, problem solving is seen as independent of, and isolated from, the development of core mathematical ideas, understandings, and processes. Despite these decades of research and associated curriculum development, it seems that students' problem solving abilities still require substantial improvement especially given the rapidly changing nature of today's world (Kuehner & Mauch, 2006; Lesh & Zawojewski, 2007; Lester & Kehle, 2003).

This current state of affairs has not been helped by the noticeable decline in the amount of problem solving research that has been conducted in the past decade. A number of factors have been identified as contributing to this decline. These include the discouraging cyclic trends in educational policy and practices, limited research on concept development and problem solving, insufficient knowledge of students' problem solving beyond the classroom, the changing nature of the types of problem solving and mathematical thinking needed beyond school, and the lack of accumulation of problem solving research (Lesh & Zawojewski, 2007). Before considering each of these contributing factors, we offer an overview of research on mathematical problem solving over the past 50 years.

A BRIEF SKETCH OF FIFTY YEARS OF RESEARCH ON MATHEMATICAL PROBLEM SOLVING

In mathematics education, research on problem solving has focused primarily on *word problems* of the type emphasized in school textbooks or tests – where "problems" are characterized as activities that involve *getting from givens to goals when the path is not obvious.* With such situations in mind, Polya's book *How to Solve It* (1945) introduced the notion of *heuristics* – such as *draw a picture*, *work backwards*, *look for a similar problem*, or *identify the givens and goals* (later referred to as *strategies* by mathematics educators) – which mathematics education researchers immediately recognized to be useful for generating after-the-fact descriptions of past behaviors for many expert problem solvers. But, even for less experienced problem solvers, these same heuristics also were expected to provide useful answers to the question: *"What should I do when I'm stuck?"*

Unfortunately, for reasons we describe briefly in this, our first of two papers for ICME 11, the past 50 years of research have not provided validation for these latter expectations. Nonetheless, some hope remains! Most past research has leaped ahead to investigate the questions: (a) *Can Polya-style heuristics be taught?* (b) *Do learned heuristics/strategies have positive impacts on students' competencies*? There exists almost no research that has provided useful *operational definitions* to answer more fundamental questions such as: (a) *What does it mean to "understand" Polya-style heuristics?* (b) *How (and in what ways) do these understandings develop?* (c) *What is the nature of primitive levels of development?* (d) *How can development be reliably observed, documented, and measured (or assessed)?* Until researchers develop useful responses to these latter two questions, it is not reasonable to expect significant progress to be made on the former two questions.

In spite of the apparent *face validity* of Polya's heuristics, Begel's (1979) comprehensive review of the research literature in mathematics education concluded that there was little evidence to support the claim that general processes that experts use to describe their past problem solving behaviors also should provide prescriptions to guide novices' next-steps. Similarly, Silver's (1985) assessment of the literature on problem solving concluded that, even in studies where some successful learning has been reported, transfer of learning has been unimpressive. Furthermore, successes generally occurred only when world-class teachers taught long and complex courses in which the size and complexity of the "treatments" made it unclear *why* performance improved. Perhaps, suggested Silver, improvements in problem solving performance simply resulted from students learning relevant mathematics concepts - rather than from learning problem solving strategies, heuristics, or problem solving processes!

Similar conclusions again were stated in the NCTM's 1992 *Handbook for Research on Mathematics Teaching and Learning* (Grouws, 1992), where Schoenfeld's (1992) chapter on problem solving concluded that attempts to teach students to use Polya-style heuristics and processes generally had not proven to be successful. However, Schoenfeld went on to suggest that one reason for this lack of success might be because many of Polya's heuristics appear to be descriptive but not prescriptive. That is, most are really just names for large categories of processes rather than being well defined processes in themselves. Therefore, in an attempt to go beyond "descriptive power" to achieve "prescriptive power," Schoenfeld suggested that problem solving research and teaching should: (a) Help students develop larger numbers of more *specific problem solving strategies* that link more clearly to specific classes of problems, (b) Teach *metacognitive strategies*[1] so that students learn when to use their problem solving strategies and content knowledge, and (c) Develop ways to improve students' beliefs about the nature of mathematics, problem solving, and their own personal competencies.

Unfortunately, ten years after Schoenfeld's proposals were made, Lester and Koehle (2003) again reviewed the literature and again concluded that research on problem solving still had little to offer to school practice. One explanation for this lack of success appeared to be because Schoenfeld's proposal simply moved the basic shortcoming of Polya's heuristics to a higher level. That is, regardless of whether attention focuses on Polya-style heuristics or on Shoenfield-style metacognitive processes or beliefs, short lists of descriptive processes or rules tend to be too general to have prescriptive power. Yet, longer lists of prescriptive processes or rules tend to become so numerous that knowing when to use them becomes the heart of what it means to understand them. This shortcoming tends to be exacerbated by the fact that regardless of whether attention focuses on Polya's heuristics or on Schoenfeld's metacognitive processes or beliefs, virtually all such processes and rules are counterproductive in some situations. For example, even the seemingly-sensible admonition for students to *carefully plan-monitor-assess their work* tends to be explicitly set aside during periods of productive "brainstorming" during initial stages of solving complex problems. In fact, the defining characteristic of brainstorming is that problem solvers are supposed to rapidly generate a diverse collection of ideas – by temporarily avoiding criticism, assessment, and concerns about long-range implications. So, knowing when and why to use such techniques emerges as one of the most important parts of what it means to understand them.

In response to such conclusions about the state of problem solving research, Schoenfeld's plenary address for the 2007 *NCTM Research Pre-session* proposed another embellishment of his same basic theory. The heart of his recommendation was that researchers should focus on something that we might call *meta-meta-cognitive processes* – or rules which are expected to operate on lower-level metacognitive processes, heuristics, strategies, knowledge, or skills. But again, just as in the case of the beliefs and metacognitive processes that Schoenfeld proposed fifteen years earlier, meta-meta-cognitive processes were described as being explicitly executable rules (e.g., cost-benefit rules which operate on lower-level rules). Consequently, it is unclear why meta-metacognitive rules should be expected to avoid the same shortcomings that were associated with past notions of heuristics, strategies, or meta-cognitive processes. That is, short lists of descriptive rules lack prescriptive power, and longer lists of prescriptive rules involve knowing when and why to use them.

Lesh and Zawojewski (2007) concluded that, when a field of research has experienced more than 50 years of failure using continuous embellishments of rule-governed conceptions of problem solving competence, perhaps the time has come to consider other options – and to re-examine foundation-level assumptions about what it means to understand mathematics concepts and problem solving processes. In particular, it is time to re-examine foundation-level assumptions about what it means to understand a small number of big ideas in elementary mathematics. One alternative is to use theoretical perspectives and accompanying research methodologies that we call *models & modeling perspectives* (MMP) on mathematics problem solving, learning, and teaching (Lesh & Doerr, 2003). But, before describing

[1] Metacognitive processes are processes that operate of lower-order knowledge or abilities.

relevant aspects of MMP, we briefly identify some of the major reasons why past problem solving research has produced so little success.

LIMITING FACTORS IN ROBLEM-SOLVING RESEARCH

Pendulum Swings Fuelled by High-Stakes Testing

Over the past several decades, we have seen numerous cycles of pendulum swings between a focus on problem solving and a focus on "basic skills" in school curricula. These approximately 10-year cycles, especially prevalent in the USA but also evident in other nations, appear to have brought few knowledge gains with respect to problem solving development from one cycle to the next (Lesh & Zawojewski, 2007). Over the past decade or so, many nations have experienced strong moves back towards curricula materials that have emphasized basic skills. These moves have been fuelled by high-stakes national and international mathematics testing, such as PISA (Programme for International Student Assessment: http://www.pisa.oecd.org/) and TIMMS (Third International Mathematics and Science Study: http://timss.bc.edu/timss2003i/intl_reports.html).

These test results have led many nations to question the substance of their school mathematics curricula. Indeed, the strong desire to lead the world in student achievement has led several nations to mimic curricula programs from those nations that score highly on the tests, without well-formulated plans for meeting the specific needs of their student and teacher populations (Sriraman & Adrian, 2008). This teaching-for-the test has led to a "New Push for the Basics" as reported in the New York Times, November 14, 2006. Unfortunately, these new basics are not the basics needed for future success in the world beyond school, as we indicate later. With this emphasis on basic skills, at the expense of genuine real-world problem solving, the number of articles on research in problem solving has declined. What is needed is research that explores students' concept and skill development as it occurs through problem solving.

Limited Research on Concept Development and Problem Solving

As we discuss in our second paper, relationships are unclear between concept development and the development of problem solving competencies (Lester & Charles, 2003; Schoen & Charles, 2003). One shortcoming of past problem solving research is that it has not been clear how concept development is expected to interact with the development of relevant problem solving heuristics, beliefs, dispositions, or processes. In fact, in many curriculum standards documents (e.g., NCTM, 2000, 2008 http://standards.nctm.org/document/chapter3/prob.htm), problem solving tends to be listed as the name of a chapter-like topic similar to algebra, geometry, or calculus. In other words, the implicit assumption is conveyed that problem solving ability is expected to increase by: (a) first, mastering relevant concepts, (b) second, mastering relevant problem solving heuristics, strategies, beliefs, dispositions, or processes, and (c) third, learning to put these concepts and processes together to solve problems. Consequently, when such assumptions are coupled with the flawed belief that students must first learn concepts and processes as abstractions before they can put them together and use them in "real-life" problem solving situations, problem solving tends to end up never getting taught at all in many classrooms. So, one of the most critical challenges for future problem solving research is to clarify the nature of relationships that should exist between concept development and the development of problem solving competencies.

Limited Knowledge of Students' Problem Solving Beyond the Classroom

As we have highlighted, problem solving is a complex endeavor involving, among others, mathematical content, strategies, thinking and reasoning processes, dispositions, beliefs, emotions, and contextual factors. Future studies of problem solving need to embrace the complexity of problem solving as it occurs in school and beyond, as we discuss later. However, to date, most research on problem solving has not really addressed students' problem solving capabilities beyond the classroom—we need to know why students have difficulties in applying the mathematical concepts and abilities (that they presumably have learned in school) outside of school—or in other classes such as those in the sciences. To assist us here we need more interdisciplinary problem solving experiences that mirror problem solving beyond the classroom (English, in press). For example, experiences that draw upon the broad field of engineering provide powerful links between the classroom and the real world, enabling students to apply their mathematics and science learning to the solution of authentic problems (Kuehner & Mauch, 2006).

Changing Nature of the Types of Problem Solving and Mathematical Thinking needed beyond School

Today, experts outside of schools consistently emphasize that new technologies for communication, collaboration, and conceptualization have led to significant changes in the kinds of mathematical thinking that are needed beyond school—and to significant changes in the kinds of problem solving situations in which some form of mathematical thinking is needed. For example, in just a few decades, the application of mathematical modeling to real-world problems has escalated. Traffic jams are modeled and used in traffic reports; the placement of cell-phone towers is based on mathematical models involving 3-D topography of the earth; and the development of internet search engines is based on different mathematical models designed to find new and more efficient ways to conduct searches. Unfortunately, the types of problems students meet in the classroom are often far removed from reality—we need to redress this state of affairs as we consider fresh perspectives on problem solving in the curriculum.

Research on problem solving beyond school also suggests that, although professionals in mathematics-related fields draw upon their school learning, they do so in a flexible and creative manner, unlike the way in which they experienced mathematics in their school days (Gainsburg, 2006; Hall, 1999; Hamilton, 2007; Noss, Hoyles, & Pozzi, 2002; Zawojewski & McCarthy, 2007). Furthermore, problem solvers beyond the classroom often are not isolated individuals but instead are teams of diverse specialists (Hutchins, 1995a, 1995b; Sawyer, 2007). These specialists often offload important aspects of their thinking using powerful technology-based tools which make some functions easier (such as information storage, retrieval, representation, or transformation) but which make others far more complex and difficult (such as interpretation and communication). So, relevant knowledge and abilities tend to be distributed across a variety of tools, and across individuals within groups. Critical abilities often are those associated with the mathematics of description, explanation, and communication at least as much as the mathematics of computation and deduction (Lesh, Middleton, Caylor & Gupta, 2008), and progress tends to resemble the evolution of a community or interacting organisms – rather than movement along a path (Lesh & Yoon, 2004). Unfortunately, research on mathematical problem solving has not kept pace with the rapid changes in the mathematics and problem solving needed beyond school.

Lack of Accumulation of Problem solving Research

As we also discuss in our second paper, there has been a lack of accumulation of problem solving research. Failed or flawed concepts or conjectures have continued to be recycled or embellished – with no significant changes being made in the underlying theoretical perspectives. Mathematics education researchers have generally avoided tasks that involve developing critical tools for their own use. Unlike their counterparts in more mature sciences (physics, chemistry, biology), where some of the most significant kinds of research often involve the development of tools to reliably observe, document, or measure the most important constructs, mathematics educators have developed very few tools for observing, documenting, or measuring most of the understandings and abilities that are believed to contribute to problem solving expertise. We return to this concern later in this paper.

Furthermore, partly because operational definitions and tools have not been developed for most constructs that have been considered important in problem solving development, there is a tendency to repeatedly elaborate on or recycle apparently failed or flawed concepts. For example, the use of Polya-style heuristics, problem solving strategies, and various metacognitive and meta-metacognitive processes (Schoenfeld, 2007) is an example of continuous embellishment of a theory that focuses on explicitly learned rules. In our second paper, we extend our discussion on theory development and explore alternative research methodologies for advancing the field.

ADVANCING THE FIELD OF PROBLEM SOLVING RESEARCH AND CURRICULUM DEVELOPMENT

Although we have highlighted some of the issues that have plagued problem solving research, there are emerging signs that the situation is starting to improve. We believe the pendulum of change is beginning to swing back towards problem solving on an international level, providing impetus for new perspectives on the nature of problem solving and its role in school mathematics (Lester & Kehle, 2003). For example, a number of Asian countries have recognized the importance of a prosperous knowledge economy and have been moving their curricular focus toward mathematical problem solving, critical thinking, creativity and innovation, and technological advances (e.g., Maclean, 2001; Tan, 2002). In refocusing our attention on problem solving and how it might become an integral component of the curriculum rather than a separate, often neglected, topic we explore the following issues:

- What is the nature of problem solving in various arenas of today's world?
- What future-oriented perspectives are needed on the teaching and learning of problem solving including a focus on mathematical content development through problem solving?
- How can studies of problem solving expertise contribute to theory development that might guide the design of worthwhile learning experiences?
- Why is a models and modeling perspective a powerful alternative to existing views on problem solving?

The Nature of Problem Solving in Today's World

Concerns have been expressed by numerous researchers and employer groups that schools are not giving adequate attention to the understandings and abilities that are needed for success beyond school. For example, potential employees most in demand in mathematics/science related fields are those that can (a) interpret and work effectively with complex systems, (b) function efficiently and communicate meaningfully within diverse teams of specialists, (c) plan, monitor, and assess progress within complex, multi-stage projects, and (d) adapt quickly to continually developing technologies (Lesh, in press). Research indicates that such employees draw effectively on interdisciplinary knowledge in solving problems and communicating their findings. Furthermore, although they draw upon their school learning, these employees do so in a flexible and creative manner, often creating or reconstituting mathematical knowledge to suit the problem situation, unlike the way in which they experienced mathematics in their school days (Gainsburg, 2006; Hamilton, 2007; Lesh, in press; Zawojewski & McCarthy, 2007). In fact, these employees might not even recognize the relationship between the mathematics they learned in school and the mathematics they apply in solving the problems of their daily work activities.

Identifying and understanding the differences between school mathematics and the workplace is critical in formulating a new perspective on problem solving. As we address later, one of the notable findings of studies of problem solving beyond the classroom is the need to master mathematical modeling. Many new fields, such as nanotechnology, need employees who can construct basic yet powerful constructs and conceptual systems to solve the increasingly complex problems that confront them. Being able to adapt previously constructed mathematical models to solve emerging problems is a critical component here.

Future-Oriented Perspectives on the Teaching and Learning of Problem Solving

We have argued that future-oriented perspectives on problem solving should transcend current school curricula and national standards and should draw upon a wider range of research across disciplines (English, 2008; Lesh, in press). Most research on problem solving has commenced with the assumption that the researchers already possess clear and accurate understandings about what it means to "understand" problem solving. This is not necessarily the case, as we have indicated (e.g., retrospective descriptions of observed problem solving do not necessarily provide useful forward-looking prescriptions for what problem solvers should do as "next steps" during problem solving sessions).

A critical component of any agenda to advance the teaching and learning of problem solving is the clarification of the relationships and connections between the development of mathematical content understanding and the development of problem solving abilities, as we have emphasized earlier in this paper. If we can clarify these relationships we can inform curriculum development and instruction on ways in which we can use problem solving as a powerful means to develop substantive mathematical concepts. In so doing, we can provide some alternatives to the existing approaches to teaching problem solving. These existing approaches include instruction that assumes the required concepts and procedures must be taught first and then practiced through solving routine "story" problems that normally do not engage students in genuine problem solving (primarily a content-driven perspective). Another existing approach, which we have highlighted earlier, is to present students with a repertoire of problem solving heuristics/strategies such as "draw a diagram," "guess and check," "make a table" etc. and provide a range of non-routine problems to which these strategies can be applied (primarily a problem solving focus). Unfortunately, both these approaches treat problem solving as independent of, or at least of secondary importance to, the concepts and contexts in question.

A powerful alternative to these approaches is one that treats problem solving as integral to the development of an understanding of any given mathematical concept or process. This perspective (Lesh & Zawojewski, 2007) also reflects the recognition that the problem solving of novices and experts differs in ways that go beyond their observed behaviors, that is, what they actually *do* in solving a problem. Novices and experts *see* (interpret and re-interpret) problem situations differently—experts focus on the *underlying structural features* of a problem situation so for them, problem solving involves an interplay between problem structure (content) and problem solving processes. We continue this discussion in the next section.

Studies of Problem solving Expertise and their Contributions to Theory Development

The seminal work of Krutetskii (1976) has shown how gifted mathematics students have a repertoire of ideas, strategies, and representations that seem to be organized into a highly sophisticated network of knowledge, equipping them with powerful ways to approach problem solving situations. As noted above, experts readily perceive the underlying structures of problem situations, project ahead to remove unnecessary steps in the solution process, and are able to generalize broadly. When we explore expert problem solving beyond the classroom, we see other factors that play a key role. For example, the knowledge of experts in workplace environments that require heavy use of mathematics tends to be more organized around the mathematics of the *situation* than around general problem solving strategies or traditional mathematical topics (e.g., Gainsburg, 2006; Hall, 1999).

Although such studies have provided rich insights into how experts perform in given problem solving situations, they do "not guarantee that one is studying the experts at what actually makes them experts" (Lester & Kehle, 2003, p. 504). In other words, how do experts become experts? *We need new studies on the nature and development of expertise*—how expertise evolves within episodes of problem solving and over many experiences. Presumably, students' understandings of problem solving are not so different from their understandings of other aspects they are to learn in mathematics. For example, students' understandings of problem solving heuristics probably develop. And, development should be able to be traced. So, we need more studies about problem solving that are similar to the studies that mathematics educators have conducted about the development of concepts and abilities in topic areas such as early number concepts, rational number concepts, early algebra concepts, and so on.

Rather than just describe the behavior we observe as experts solve problems, we need to know how they interpret the problem situations, how they mathematize them, how they quantify them, how they operate on quantities, and so on (Lesh & Zawojewski, 2007). Furthermore, we need to look beyond the assumption that experts initially learn content, then acquire problem solving strategies, and then learn ways to apply the mathematics and strategies they have developed. As Zawojewski and Lesh (2003) and others have argued, the development of problem solving expertise appears as a synergistic, holistic development of varying degrees of mathematical content, problem solving heuristics/strategies, higher-order thinking, and affect—all of which are situated in particular contexts.

Theory Development: A Models and Modeling Perspective (MMP) on the Development of Problem Solving in and beyond the Classroom

Before we explore theory development, we need to offer a more appropriate definition of problem solving, one that does not separate problem solving from concept development as it occurs in real-world situations beyond the classroom. We adopt here the definition of Lesh and Zawojewski (2007):

> A task, or goal-directed activity, becomes a problem (or problematic) when the "problem solver" (which may be a collaborating group of specialists) needs to develop a more productive way of thinking about the given situation (p. 782).

Thinking in a productive way requires the problem solver to interpret a situation mathematically, which usually involves progression through iterative cycles of describing, testing, and revising mathematical interpretations as well as identifying, integrating, modifying, or refining sets of mathematical concepts drawn from various sources (Lesh & English, 2005; Lesh & Zawojewski, 2007). These processes are the rudiments of mathematical modeling. Seeing problem solving from a models and modeling perspective (MMP) contrasts with the traditional definition of problem solving as searching for a way to progress from the "givens" to the "goals." Rather, from a models and modeling perspective, problem solving involves iterative cycles of understanding the givens and the goals of a problem. In the remainder of this section we highlight some of the key features of problem solving from a models and modeling perspective.

1. When the solution to a problem involves the development of a model (or artifact or conceptual tool), and when the underlying conceptual systems are expressed in forms that can be examined and assessed by students themselves, solutions tend to involve *sequences of iterative express-test-revise cycles* similar to the kind that are involved in the first-, second-, and nth-drafts in the development of other kinds of symbolic or graphic descriptions of situations. Furthermore, if the underlying conceptual system is one that Piaget-inspired researchers have investigated, then the modeling cycles that problem solvers go through during a single 60-90 minute problem solving session are often strikingly similar to the stages of development that Piagetians have documented over time periods of several years. Consequently, we have sometimes referred to such sessions as *local conceptual development sessions* (Lesh & Harel, 2003) – because students' thinking often evolves through several stages similar to those recognized by the Piagetians during a single 60-90 minute episode.

2. When significant conceptual adaptations occur within a single problem solving session, researchers are able to go beyond observing sequential states of knowledge to also *directly observing processes that lead from one state to another*. And, such observations have made it clear that it is seldom appropriate to think of solution processes as activities in which students connect previously-mastered-but-disconnected concepts and processes. Nor do solutions involve movement along a path which is formed by linking together concepts, processes, facts, and skills. Instead, problem solvers' early interpretations tend to involve a collection of partly-overlapping-yet-undifferentiated partial interpretations of different aspects of the situations of conceptual systems. So, regardless of whether the problem solver is an individual or a group, *model development tends to involve gradually sorting out, clarifying, revising, refining, and integrating conceptual systems that are at intermediate stages of development.*

3. When solutions to problems involve the development of mathematically significant artifacts or tools, when the underlying design is an important part of the product that is designed to help solve the problem, and when the product needs to be powerful (for the specific situation in which it was first created), sharable (with other people), and reusable (in situations different to the one in which it was first created), the *knowledge and abilities that are embodied*[2] *in these products tend to be generalizable and transferable.*

4. *Heuristics* that are intended to help problem solvers make productive adaptations to existing ways of thinking tend to be significantly different from heuristics that are intended to help problem solvers figure out what to do when they are stuck (with no apparent concepts available). Their functions tend to have far less to do with helping students know what to do next, and have far more to do with *helping them interpret the situation* (including alternative ways of thinking about givens, goals, personal competencies, and "where they are" in solution processes). Furthermore, such heuristics often function tacitly rather than as explicitly executed rules. So, learning them is similar to situations in which athletes or performing artists analyze videotapes of their own performances (or those of others). It is useful to develop languages (interpretation systems and conventions for making interpretations) for describing these past performances. But, such languages usually are not intended to give rise to prescriptive rules about what to do at specific points in future

[2] Here, we use the term "embodiment" in the way used by Zoltan Dienes who introduced the notion of concrete embodiments of mathematical concepts (Dienes, 1960).

performances. *Instead, the language and imagery tends to be aimed mainly at helping students make sense of things during future performances. In other words, they are aimed mainly at the development of more powerful models.*

5. When problem solving involves model development, *heuristics and metacognitive processes tend to evolve in ways that are quite similar to the dimensions of development that apply to other types of concepts or abilities that mathematics educators have studied.* For example, Vygotsky's (1978) concept of internalizing external functions often results in early understandings of heuristics that are distinctly social in character. So, instead of "looking at a similar problem" students may find it more useful to think of themselves as "looking at the same problem from another point of view" (and to be aware of the fact that one's current point of view is not the only possible point of view).

6. In fields like engineering it is considered to be "common knowledge" that realistic solutions to realistically complex problem solving situations nearly always need to integrate concepts and procedures drawn from a variety of textbook topic areas or theories. Likewise, when students develop realistically useful models (or other conceptual tools) for making sense of realistically complex "real-life" situations, *they often need to integrate ideas and procedures drawn from more than a single textbook topic area* (measurement, geometry, probability, statistics, and algebra). One reason for this is because useful solutions often involve *trade-offs* involving conflicting goals associated with multiple agents. These goals may involve low costs but high quality, or low risk but high gain, or rapid but thorough development. The models that are produced are "chunks of knowledge" that represent inherently connected ideas that need to be unpacked in follow-up teaching and learning activities. Even after connected ideas are unpacked, *students' knowledge often continues to be organized around experience as much as it is organized around abstractions*.

7. When problem solvers describe or design things mathematically, they tend to do more than simply engage logical-mathematical systems; they also engage feelings, values, beliefs, and a variety of problem solving processes, facts, and skills. So, *the development of processes, skills, attitudes, beliefs, is part of the development of specific models*. Skills, attitudes, and beliefs are not developed separately in the abstract before they are connected to concepts or conceptual systems; skills, attitudes, and beliefs are engaged and developed when the relevant models are engaged. Thus, skills, attitudes, and beliefs are integral parts of relevant models.

CONCLUDING POINTS

We have argued in this paper that research on mathematical problem solving has stagnated for much of the 1990s and early part of this century. Furthermore, the research that has been conducted does not seem to have accumulated into a substantive, future-oriented body of knowledge on how we can effectively promote problem solving within and beyond the classroom. This lack of progress is mainly due to the many years of repeated elaborations of rule-governed conceptions of problem solving competence.

The time has come to consider other options for advancing problem solving research and curriculum development—we have highlighted the need to re-examine foundation-level assumptions about what it means to understand mathematics concepts and problem solving processes. One powerful alternative we have advanced is to utilize the theoretical perspectives and accompanying research methodologies of a *models & modeling perspective* (MMP) on mathematics problem solving, learning, and teaching. Our second paper elaborates further on this perspective and on the associated research methodologies.

Adopting an MMP means researchers who study students' models and modeling developments naturally utilize integrated approaches to exploring the co-development of mathematical concepts, problem solving processes, metacognitive functions, dispositions, beliefs, and emotions. These researchers also view problem solving processes developmentally, in a similar way they would in studying the development of mathematical concepts in topic areas such as early number, geometry, and algebra. In addition, the problems used are simulations of appealing, authentic problem solving situations (e.g., selecting sporting teams for the Olympic Games) and engage students in mathematical thinking that involves creating and interpreting situations (describing, explaining, communication) at least as much as it involves computing, executing procedures, and reasoning deductively.

REFERENCES

Anderson, J. R., Boyle, C. B., & Reiser, B. J. (1985). Intelligent tutoring systems. *Science*, 228, 456-462.

Begle, E. G. (1979). Critical Variables in Mathematics Education. Washington D. C.: the Mathematics Association of America and the National Council of Teachers of Mathematics.

Charles, R. & Silver, E., (1988). *The teaching and assessing of mathematical problem solving*. Reston, VA: National Council of Teachers of Mathematics.

Dienes, Z. (1960). *Building up Mathematics.* London: Hutchins Education Ltd.

English, L. D. (2007). Complex systems in the elementary and middle school mathematics curriculum: A focus on modeling. In B. Sriraman (Ed.), Festschrift in Honor of Gunter Torner. *The Montana Mathematics Enthusiast*, (pp. 139-156). Information Age Publishing.

English, L. D. (2008). Mathematical modeling: Linking Mathematics, Science, and the Arts in the Elementary Curriculum. In B Sriraman, C. Michelsen, & A. Beckmann, & V. Freiman (Eds.), Proceedings of The Second International Symposium on Mathematics and its Connections to the Arts and Sciences (MACAS2, pp. 5-36). University of Southern Denmark Press.

English, L. D. (In press, 2009). Promoting interdisciplinarity through mathematical modelling. *ZDM: The International Journal on Mathematics Education, 41*(1).

Gainsburg, J. (2006). The mathematical modeling of structural engineers. *Mathematical Thinking and Learning,* 8(1), 3-36.

Grouws, D., (1992). (Ed.), *Handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (pp. 334-370). New York, NY: Macmillan Publishing Co.

Hall, R. (1999). *Case studies of math at work: Exploring design-oriented mathematical practices in school and work settings* (NSF Rep. No. RED-9553648), Arlington, VA: National Science Foundation.

Hamilton, E. (2007). What changes are needed in the kind of problem solving situations where mathematical thinking is needed beyond school? In R. Lesh, E. Hamilton, & J. Kaput (Eds.), *Foundations for the future in mathematics education* (pp. 1-6). Mahwah, NJ: Lawrence Erlbaum.

Hutchins, E. (1995a) *Cognition in the Wild.* Cambridge, MA: MIT Press.

Hutchins, E. (1995b) How a cockpit remembers its speeds. *Cognitive Science.* 19, 265-288.

Krutetskii, V. (1976). *The psychology of mathematical abilities in school children.* Chicago, IL: University of Chicago Press.

Kuehner, J. P., & Mauch, E. K. (2006). Engineering applications for demonstrating mathematical problem solving methods at the secondary education level. *Teaching Mathematics and its Applications, 25*(4), 189-195.

Lesh, R. & Doerr, H. (2003). Foundation of a Models and Modeling Perspective on Mathematics teaching and Learning. In R.A. Lesh & H. Doerr (Eds.), *Beyond Constructivism: A models and modeling perspective on mathematics teaching, learning, and problem solving.* 9-34. Mahwah, NJ: Erlbaum.

Lesh, R. & English, L. D. (2005). Trends in the evolution of models and modeling perspectives on mathematical learning and problem solving. In H. Chick & J. Vincent (Eds.), Proceedings of the 29th Annual Conference of the International Group for the Psychology of Mathematics Education. (pp. 192-196). University of Melbourne.

Lesh, R. & Harel, G. (2003). Problem solving, modeling, and local conceptual development. *International Journal of Mathematics Thinking and Learning, 5,* 157-189.

Lesh, R. & Yoon, C., (2004) Evolving communities of mind-In which development involves several interacting and simultaneously developing strands. *Mathematical Thinking and Learning, 6*(2), 205-226.

Lesh, R. & Zawojewski, J. S. (2007). Problem solving and modeling. In F. Lester (Ed.). *The Second Handbook of Research on Mathematics Teaching and Learning*. (pp. 763-804). Charlotte, NC: Information Age Publishing.

Lesh, R. (In press). Directions for future research and development in engineering education. In J. Zawojewski, H. Diefes-Dux, & K. Bowman (Eds.), Models and modeling in Engineering Education: Designing experiences for all students. Rotterdam: Sense Publications.

Lesh, R., Middleton, J., Caylor, E., & Gupta, S., (2008) A science need: Designing tasks to engage students in modeling complex data. *Educational studies in Mathematics*, *68*(2), 113-130.

Lester, F. K. & Charles, R. I. (Eds.) (2003). Teaching mathematics through problem solving: PreK - 6. Reston, VA: National Council of Teachers of Mathematics.

Lester, F. K. & Kehle, P. E, (2003). From problem solving to modeling: The evolution of thinking about research on complex mathematical activity. In R. A. Lesh & H. M. Doerr (Eds.), Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching (pp. 501-518). Mahwah, NJ: Lawrence Erlbaum Associates.

Lester, F. K., Garofalo, J., & Kroll, D. L. (1989). Self-confidence, interest, beliefs, and metacognition. Key influences on problem solving behavior. In D. B. McLeod & V. M. Adams (Eds.), Affect and mathematical problem solving: A new perspective (pp. 75-88). New York: Springer-Verlag.

Maclean, R. (2001). Educational change in Asia: An overview. *Journal of Educational Change 2*, 189-192.*edies.* London: World Scientific, 2003.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

NCTM Math Standards (2008) http://standards.nctm.org/document/chapter3/index.htm Accessed: 22 May 2008

Noss, R., Hoyles, C., & Pozzi, S. (2002). Abstraction in expertise: A study of nurses' conceptions of concentration. *Journal for Research in Mathematics Education, 33*(3), 204-229.

Petroski, H. (2003). Early education. *American Scientist*, 91, 206-209.

Polya, G. (1945). *How to solve it*. Princeton, NJ: Princeton University Press.

Sawyer, R. K. (2007). *Group Genius: The Creative Power of Collaboration*. New York: Basic Books.

Schoen & Charles, (2003). (Eds.), *Teaching mathematics through problem solving: Grades 6-12.* Reston, VA. National Council of Teachers of Mathematics.

Schoenfeld, A. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (pp. 334-370). New York, NY: Macmillan Publishing Co.

Schoenfeld, A. (2007). *Problem Solving Reconsidered: Toward a Theory of Goal-Directed Behavior.* Presentation given at the 2007 NCTM national conference pre-session. Address delivered in Atlanta, GA, USA. 19 March 2007.

Silver, E.A. (1985). Research on teaching mathematical problem solving: Some under represented themes and needed directions. In E. A. Silver (ed.), Teaching and learning mathematical problem solving. Multiple research perspectives (pp. 247-66). Hillsdale, NJ: Lawrence Erlbaum Associates.

Sriraman, B., & Adrian, H. (2008). A critique and response to multicultural visions of globalization. *Interchange 39*(1), 119-130.

Tan, J. (2002). Education in the twenty-first century: Challenges and dilemmas. In D. da Cunha (Ed.) *Singapore in the new millennium: Challenges facing the citystate.* (p. 154-186). Singapore: The Institute of Southeast Asian Studies.

Vygotsky, L.S. (1978). *Mind and society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press.

Zawojewski, J. & Lesh, R. (2003). A Models and Modeling Perspective on Problem Solving Strategies In R.A. Lesh & H. Doerr (Eds.), *Beyond Constructivism: A models and modeling perspective on mathematics teaching, learning, and problem solving.* 9-34. Mahwah, NJ: Erlbaum.

Zawojewski, J. & McCarthy, L. (2007). Numeracy in practice. *Principal Leadership, 7*(5), 32-38

Zawojewski, J., Hjalmarson, J. S., Bowman, K., & Lesh, R. (In press). A modeling perspective on learning and teaching in engineering education. In J. Zawojewski, H. Diefes-Dux, & K. Bowman (Eds.), Models and modeling in engineering education: Designing experiences for all students. Rotterdam: Sense Publications.