A TECHNOLOGY-BASED INVESTIGATION OF UNITED STATES HIGH SCHOOL STUDENT MATHEMATICAL PROBLEM SOLVING

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Summary

The purpose of this paper is to discuss an investigation of students' behavior during mathematical problem solving that may provide new ways for teachers to help students think about mathematics. This study explored 122 high school students' problem solving strategies, using technology that enabled the researcher to track and model in detail the steps students used to answer mathematical problems. This paper reports findings explaining the different problem solving strategies and metacognitive approaches that students used to solve various types of mathematics items. Based on study findings, the paper also recommends strategies teachers can use to help students become more metacognitively aware of their mathematical thinking. Finally, the paper describes the technology used to document and analyze student problem solving behaviors.

Introduction describing the study and research questions

A major challenge for teachers of mathematics is understanding what students know and what misconceptions deter them from solving problems correctly. Teachers can infer that students who are higher achieving (as measured by grades and test scores) understand more than do students who are lower achieving, but that inference merely allows us to stratify them, not to deeply understand how they are engaging with the mathematics.

The purposes of this study, then, are 1) to identify the strategies that high school students use when solving mathematics problems so as to better understand the processes they use and 2) to uncover some potential reasons females underperform in mathematics compared to males. This study enables a more detailed understanding of student test-taking behavior by providing a more authentic look at what students do before they choose a final answer. Ultimately, determining what enables higher-performing students to respond correctly can inform new ways of conceptualizing instruction.

Theoretical framework

The need to improve students' mathematical problem solving capacity in the United States can be seen in the performance of U.S. 15-year-olds on the Program for International Student Assessment [PISA] test, in which the U.S. ranked 24th out of 29 developed nations in mathematics literacy and problem solving (Augustine, 2007, OECD, 2004). One reason for the poor performance of U.S. students may be that these students are not provided with instruction that successfully integrates content learning with experiences that foster their understanding of problem solving processes (NRC, 2006).

Problem Solving

To improve mathematics instruction in problem solving, educators must first better understand specifically what it is that students do and understand when they problem solve. Strengthening students' mathematical problem solving requires that teachers actively engage students in a complex mathematical task in which students construct their understanding of mathematics in nonroutine ways. Performing such a task, however, does not always imply a thorough understanding of that task (Stevens, et al., 2004; 2005; 2006). Students can perform mathematical steps without making the connections necessary to transfer or apply that knowledge in a different context (Lajoie, 2003). Rather than simply performing steps, to solve a problem successfully and understand what they are doing, students need to have knowledge of the content and a strategy to solve the problem. They must monitor their progress through to the solution and must be motivated to tackle the problem and solve it (Marshall, 1995). Instructional approaches that emphasize that students take responsibility for their own learning—that is, approaches that provide them with opportunities to choose, guide, and assess their own task activities and progress—are key to student success in problem solving and in making the connections necessary to apply their problem solving skills in new contexts (Inquiry Synthesis Project, 2006; NRC, 2006).

When approaching mathematics problems, students rely on various resources and types of information (Chi & Glaser, 1985; Ericsson & Simon, 1993; Schoenfeld, 1988). These resources and types of information form the frameworks that students use to interpret and solve different items. Ideally, students are able to identify and interpret a problem sufficiently well to choose the correct framework for solving it, resulting in a correct response. In reality, however, many students sometimes do not know which framework to choose, or choose inappropriate frameworks, which results in inconsistent patterns of correct and incorrect responses (Marshall, 1995; Tatsuoka, 1993). Furthermore, choosing an appropriate framework does not necessarily always lead to a correct answer, because computational errors can also lead to an incorrect response (Tatsuoka, 1990). Thus, trying to explicate students' problem-solving processes requires that teachers and researchers look beyond their correct and incorrect answers and undertake instead a detailed, empirical investigation of how students organize information.

Every mathematics problem contains a host of concepts that can be linked to a group of specific steps that must be followed to successfully solve the problem. When students choose to follow certain steps in a particular order, they are demonstrating a pathway for organizing information to solve that problem. By assembling detailed information on students' responses to multiple problems, researchers can trace the steps that students took to solve each problem and evaluate how well their approaches worked on different types of items—at the individual student level and across various groups of students (e.g., by classrooms). Most students refine their problem-solving strategies over time—which is consistent with models of skill acquisition (Ericsson, 2004)—gradually using fewer steps and eventually settling on a preferred approach (Stevens, Soller, Cooper, & Sprang, 2004; Stevens & Soller, 2005). Researchers can analyze student self-regulation and self-monitoring of strategies by investigating these steps, which will provide a better understanding of the complexity of student problem-solving performance (Hartley & Bendixen, 2001; Song & Hill, 2007).

IMMEX: Using Technology to Study Problem Solving

Given the myriad ways that students can organize information and regulate the steps they take to solve problems, the project of truly understanding what students are thinking and doing as they work problems can seem insurmountable. Advances in technology, however, have provided one way to get inside students' heads, as it now enables us to track in detail the actual steps students take to solve problems and the time they spend on each step (Hartley & Bendixen, 2001).

One pioneering technology that enables better understandings of student thinking is the Integrated (now Interactive) Multi-Media Exercises (IMMEX) program, which draws on both case-based (Schank, 1990) and production system (Newell & Simon, 1972) models of problem solving. Such a tool gives a more qualitative look at how students solve problems, since it captures in intricate detail the variety of approaches they can take. This captured information opens the door to a deeper understanding of the comprehensive nature of student thinking because rather than analyzing only a student's final answer, this tool allows researchers to look at each step that led to that answer.

IMMEX presents a problem to be solved in a *problem space*—that is, a space with a finite set of concepts, numbers, and equations that students must combine in order to create a solution path. Within the IMMEX problem space, various drop-down menus provide pathways the students can choose from. A problem space typically will not encompass all the combinations that students could use to solve problems, but it does provide an essential preliminary view for better estimating what they are able to do (Tatsuoka, 1990, 1995). Further, although IMMEX's simulated problem spaces are finite, they do provide enough different types of information that students with diverse math backgrounds could successfully solve the problems.

Working in IMMEX, students can assess the *problem structure*—the information needed to solve the problem—and then organize a mathematical *representation*—an arrangement of that information into a series of steps that solves the problem (Bennett, Morely, & Quardt, 1998). Most students want to arrive at an answer and will follow some process to produce one. If their chosen process leads to a wrong answer, they will probably try a different process if given a chance to try again. IMMEX software allows researchers to track all the steps—forward, backward, and sideways—that students take as they attempt to solve problems. IMMEX also records and displays the sequence of steps and the time spent on each one (Ericsson & Simon, 1993; Stevens, 1991).

Design and Methodology

This study analyzed—and compared across gender—the problem-solving strategies of 122 high school students in their junior and senior years (72 females and 50 males) on a set of SAT mathematics problems administered both on paper and in IMMEX. I also interviewed a subset of students for their feedback and impressions about solving problems online, a mode that forced them to show their steps.

Methods

To understand the specific steps that students take when tackling different kinds of mathematical problems, this study used Interactive Multi-Media Exercises (IMMEX; www.immex.ucla.edu), which consists of a library of online multimedia simulations for problem solving. IMMEX has a refined set of modeling tools for monitoring student's performance and progress (Stevens et al., 2004, Stevens & Soller, 2005; Soller & Stevens, 2007). I presented students with 25 mathematics items on the IMMEX platform. For each problem, IMMEX provided students with various menus to choose from, in a simulated problem space that is composed of a finite but representative set of concepts, numbers, and equations that students can combine to create a solution path. With these menus, students had to identify, define, and represent their steps to solve the problem. They developed reasons for choosing information that might or might not be productive in helping them find an answer (Baxter & Glaser, 1997; Stevens, et al., 2003; 2004; 2005; 2006). This case-based paradigm simulates situations with sufficient information that students with diverse experiences can successfully synthesize and thus solve the problem. IMMEX allows for detailed investigation of the steps and procedures that students use to complete a task, because all steps are documented by sequence and time spent per step.

I analyzed students' strategies to see which students planned and self-regulated their learning. I found that the more successful problem solvers tended to take time initially to identify the constraints of the problem and the steps necessary to solve it before embarking on those steps. As a result, they took fewer steps, used fewer nonproductive steps, and looked ahead by assembling their procedures before acting on them. By breaking a problem into manageable and familiar steps, successful problem solvers can chunk information into concepts and hierarchies that facilitate good problem solving (Baxter & Glaser, 1997; Paek, 2002; Siegler, 1988). This finding suggests that teachers should help students become more purposeful in (or regulate) their problem solving strategies.

Procedure

Participants first completed a retired SAT-I test from the 10 Real SATs (College Board, 1997) under standard SAT time constraints using paper and pencil. Next, students used IMMEX to solve 31 SAT-I mathematics problems that came from the same SAT-I test the students had completed for the first part of the study. Eight students also participated in a focus group to reflect on their problem solving strategies and to explain their thinking behind their steps.

Developing the IMMEX Problem Space

I created a specific problem space in IMMEX for solving each of the 31 mathematics items along with a method for using and coding the search path maps. The problem space included formulas, definitions of concepts, and a breakdown of the process for arriving at a solution. The problem space for this study was developed based on a formal task analysis the researcher had already conducted, for which students listed the steps they used to solve certain math items. Additionally, the researcher incorporated into the problem space common errors that students make in arithmetic, basic algebra, and geometry (Tatsuoka, 1990, 1995). These two elements (the formal task analysis and the incorporation of common errors) helped to determine the menus and submenus needed for the IMMEX platform in this study so that the majority of students could solve the problems using the information given (Mislevy, Yamamoto, & Anacker, 1991).

The problem space was the same for each of the 31 items in IMMEX, except for some of the submenus, which were changed to correspond with the proper substeps and the numbers and equations related to each problem. The problem space included the math concepts necessary for a correct solution as well as bugs and distractors, which were included to track where students made arithmetic errors or had misconceptions about the problem. Students could easily navigate through all the menus and submenus and still not be able to correctly solve a problem—to reach an accurate solution, they needed to know what kinds of information were pertinent and be able to order that information correctly. Within the IMMEX problem space, each problem was presented with five main menus. The problem and the five main menus were always at the top of the screen, even when students navigated through the submenus. Each main menu represented one of five math concepts: arithmetic, angles, area, perimeter, and solving equations. Clicking on one of these menus revealed a host of submenus also representing math concepts. Clicking on a submenu led to a series of equations and/or numbers. These equations/numbers were represented in expanded form, so that the student had to decide where to combine or collapse terms. The menu structure included shortcuts so that students could collapse several steps into one. For example, consider the equation $2x + 10 = 5 - 3x$. The traditional way of solving it would be first to move the numbers to one side by subtracting 10: $2x + 10 - 10 = 5 - 10 - 3x$. To get the variables on one side of the equal sign, the next move would be to add $3x$ to both sides: $2x + 3x = -5 - 3x + 3x$. Then both sides of the equation would be divided by 5: $5x/(-5) = -5/5$ to arrive at the final response of $x = -1$. In IMMEX, the menu shortcuts enabled students to collapse these three steps, computing the information in their heads so they could move to the answer in one step.

The IMMEX problem space also included common arithmetic errors students could make that were associated with the distractors offered on the paper-and-pencil test. In the problem above, for example, students could incorrectly subtract by $3x$ so that the response would be $2x - 3x = -5$, resulting in a final answer of $x = 1$. These types of mistakes were included as options in the submenus and the equation structures. The purpose of these incorrect paths was to document where students might go wrong in coming up with their final answer choice.

After completing all the work and arriving at an answer, students entered their responses after clicking the “solve problem” button. The study was structured to give students two chances to solve the problem so that they could reconsider each step they had taken and so that the researcher would have an opportunity see how students revised their steps to answer the problem correctly on the second try. The second chance also allowed students who might have made a simple arithmetic error to backtrack and correct it.

Research results

Number of Steps Taken

Using the data IMMEX collected, the number and types of steps the students took to solve each problem were analyzed. In general, the fewer steps a student had taken in attempting to solve a problem, the more likely it was that the student had solved the problem correctly—as students who unsuccessfully completed a problem tended to take more irrelevant steps that were not helpful to solving the problem. In addition, the number of steps taken differed between males and females. On average, males took two fewer steps than females did to solve a problem: males averaged four or fewer steps (M = 3.91, SD = 2.06), whereas females averaged six or more steps (M = 6.12, SD = 1.77). Even with these differences, females attempted more IMMEX items, took longer to solve each problem, and answered more problems correctly than males did; the reason for the higher success of females on these items appears to be that they verified their steps, not that they were inefficiently taking extra steps. On the paper-and-pencil SAT-I mathematics sections, however, females scored lower than did the males. The IMMEX test had no time constraints, so it may be that females performed better in the untimed situation than males did.

Informal interviews with the participants suggested that the females liked to be sure of their answers and would use any available information to verify them. They wanted their answers to be correct on the first attempt, and they took more steps and more time to ensure correctness. The interviews suggested that males, on the other hand, tended to be inclined toward an answer and would select it, knowing they had a second chance if they got it wrong. This method resulted in fewer steps and less time taken per problem. Males indicated they were also more likely to guess once they had eliminated some choices.

Amount of Time Spent per Step and per Problem

I also analyzed time spent per step and per problem. Females tended to take 2 s more per step ($M = 19$, $SD = 21$) than did males ($M = 17$, $SD = 18$), which resulted in females taking about 55 s more per problem (M = 2:06, SD = 2:51) than males (M = 1:11, SD = 1:45). These differences are statistically significant ($p < 0.01$). This difference in time spent per step, coupled with the fact that females took more steps than males did, may well help to explain why females tend to score lower on standardized mathematics tests: They are not able to complete as many problems.

Number of Attempts Made

Students were given two opportunities to solve each IMMEX problem, so I could document the changes they made in their steps from the first to the second attempt. The majority (61%) of students solved the problems correctly on the first try, and an additional 23% answered the problems correctly on their second try. The steps students took on the second tries for both correct and incorrect answers were analyzed. Students who correctly solved a problem on the second try demonstrated an orderly process in which they deliberately retraced their steps to verify their answers and more systematically regulated their steps, indicating that these students knew what they were doing but had made a small error in computation at some point. Students who did not solve the problem correctly on the second try, on the other hand, showed less organization and planning in their process.

Observing the processes used by the participants gives an inside view of how they regulated their learning as they solved problems, and suggests reasons for the performance differences documented between females and males. The amount of time and number of steps to solve each problem varied between males and females, with females taking extra steps to verify their answers and therefore taking more overall steps per problem than males. This verification process resulted in females spending more time on each problem than did males. The extra steps and time, however, paid off in the females performing slightly higher than the males on the IMMEX problems.

Discussion

The results confirmed the outcomes from previous research (e.g., Gallagher, 1990; 1992; Gallagher & De Lisi, 1994) about differences in gender in problem solving: females tended more than males to follow algorithms and verify their steps and the answer before moving on to the next problem. In the interviews, females articulated a greater need for verification in their work.

This research shows the importance of understanding in detail the steps that students take when solving mathematics problems. Tracing students' steps allows researchers to better understand how students organize information when coming up with an answer. In the present study, tracing students' steps illuminated some of the reasons that females' math test scores are typically lower than males' scores. Finding out what knowledge students possess no longer needs to be surmised only from final answers and scores, as the use of IMMEX in this study demonstrates. Researchers can and must continue to probe the processes that students employ and the knowledge they bring to bear when confronted with a mathematics problem. The more deeply that educators and researchers can analyze student thinking, the better we can measure students' competence, knowledge, and abilities—and thus the better we can design tools and practices for teachers to teach them effectively.

This study contributes to the literature on the mechanics of students' problem solving (e.g., Baxter & Glaser, 1997; Lajoie, 2003; Marshall, 2005; Stevens, et al., 2003; 2004; 2005; 2006). The findings from this study also show how technology such as IMMEX can provide two key strategies for improving mathematics teaching and learning: it can provide teachers with access to student thinking that is usually not obvious and thus enable them to modify instruction appropriately (Pellegrino, et al. 2001), and it can provide opportunities for students to reflect and fine-tune their problem solving strategies, giving them a strong context for thinking about and being successful in mathematics. A main implication of this study is for educators to find ways to increase students' metacognitive skills in mathematics so that when they participate in assessments, their performance reflects their actual understanding rather than their habitual approaches to problem solving.

The differences that the present study found in self-regulation of problem solving demonstrate the importance of teaching students how to plan their problem-solving. An underutilized strategy in instruction is having students practice using an overarching schema for new and novel problems. Other strategies include having students plan their steps before they actually begin to solve a problem, and then having them reflect on these plans and actions. In fact, a five-stage problem-solving process that is recommended in most textbooks and resources emphasizes that students should begin by reading and understanding the statement of the problem, then analyze the given facts, and then propose steps to solve the problem. These three stages are the planning phases, which should take place before students actually work the problem. The final two stages are carrying out the planned steps and then verifying the solution. On the basis of this study, a sixth stage is recommended: Teachers should provide opportunities for students to reflect on the steps they took to solve the problem. This allows students time to reflect on their previous work and, if necessary, plan a better set of steps for upcoming problems.

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