THE CONTRIBUTION OF STATISTICS IN TEACHING THE CONCEPT OF MATHEMATICAL FUNCTION

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In Brazil, the concept of mathematical function is introduced in the 8th grade, observing efforts of contextualization. However, the majority of the examples utilized at this teaching level relate to the modeling of geometrical and numerical patterns and, when they relate variables inherent to the phenomena of empirical sciences, deterministic functions are presented that make it impossible to establish any connections between the observed phenomenon and the abstract model that represents it. On the other hand, the elementary concepts of Statistics and Probabilities are presented with little relation to mathematical functions, serving only to illustrate "exotic" types of functions, without exploring sampling and variation of the data inherent to the phenomena being studied. This work discusses the potential use of the elementary concepts of Statistics and of the role of scientific thought in the formation of the concept of mathematical function, aimed at the meaningful learning of this concept.

Keywords: statistics education, mathematical function, graphs, contextualization, variation.

INTRODUCTION

In Brazil, the National Curricular Parameters for Mathematics (Brazil, 2002) highlight the interdisciplinary potentials of the concept of mathematical functions. By means of functions, one can express physical phenomena such as the variation of distance in function of time in uniform movement, biological phenomena such as the rate of photosynthesis, population increase of human beings, or bacteria, in function of time, or economic aspects such as the cost of products in function of the quantity of units, among others.

In accordance with the aforementioned parameters, the Catalogue of the National Program of Secondary Education Textbooks (Brazil, 2004) affirms that linear, quadratic, trigonometric and exponential functions are fundamental for the study of natural phenomena and can therefore favor the interdisciplinary characteristic so yearned for in the teaching of Mathematics, adding the following:

> Representation on the Cartesian plane allows one to link the properties of a function to those of the graph of said function; therefore analytic geometry can appear as a field of confluence of several concepts—function, equation, geometrical figure, etc.—which should be developed and integrated during secondary education. Finally, the treatment of topics such as increase, decrease, variation rate of function, graph inclination (sic), among others, must permeate the study of the different functions to be learned. (P. 71)

In Brazil, the concept of function is introduced in the 8th grade of Primary Education, continuing during Secondary Education. Analyzing the Mathematics textbooks, it can be observed that the authors have striven to put the concept into context, utilizing examples that relate variables connected to the biological and physical phenomena, among others (Silva U., 2007). However, they present the function in a deterministic manner, making it complicated for the learner to comprehend how to transform the empirical evidence of the "observed" phenomenon to the mathematical model.

On the other hand, it can be observed that those same textbooks—regarding the presentation of the elementary concepts of Statistics and Probabilities—fail to make the bridge between statistical concepts and the mathematical concepts that underlie the mathematical function. Nor are they able to propose real situations, within the students' reach, in order to generate empirical data; based upon that, it is possible to encounter abstract models described by the mathematical functions.

Working the "statistical functions" prior to presenting the mathematical function and the formalization thereof, on the one hand, comes into conflict with the definition of the concept of mathematical function. But, on the other hand, it can help the learner establish relations observed in the phenomena of the empirical sciences that permeate everyday life. This learning strategy can allow the student to make sense out of the subsequent mathematical modeling and to observe the regularities that are present in the phenomena studied, stimulating his/her investigative capacity.

Thus, this paper includes reflections on the potential of teaching the elementary concepts of Statistics and Probabilities in Primary Education as an element of motivation and contextualization in the teaching of mathematical functions. Its objective is to present math teachers other possibilities for introducing the concept of functions, related to the observable phenomena addressed by Statistics, in the perspective of the scientific research, contributing toward the students' scientific and civic education.

THE MATHEMATICAL FUNCTION

Mathematics is an area of knowledge that has, as a characteristic, the diversity of records, and the teaching of mathematics does not usually take this diversity into consideration, bringing about difficulties of articulation and mobilization among the different representations of a mathematical object, and consequently less apprehension of such teaching, which can reduce learning mathematics to a mechanical process (Duval, 1995).

In this context, the concept of functions and the teaching thereof constitute a key-piece in the construction of mathematical knowledge, marking the passage from arithmetic to algebra, from concrete thought to abstract thought, which requires the dominion and coordination of various symbolic representations among the phenomenon observed and the expression thereof in maternal, algebraic, tabular, graphic, and arrow-based language.

For Leinhardt, Zaslavsky and Stein (1990), algebraic and graphic representation are two systems of different symbols that are articulated in such a way that together they construct and define the concept of function, and these systems cannot be treated as isolated topics. These are communicative systems, since on one side there is a construction, and on the other side there is an organization of the mathematical ideas. Its comprehension by the learner brings challenges, since prior to this concept, the learner has dealt only with the concrete mathematical concept, and the functions and graphs are symbolic systems utilized to complement one another, demanding the use of new ideas, a unique notation, and symbolic correspondences. This, however, does not hinder an intuitive or empirical substantiation; it can serve as an interesting bridge from concrete reasoning to abstract reasoning and to reasoning within the abstractions.

However, the way in which this concept is taught can generate obstacles to its comprehension by the students. Sierpinska (1992, p. 40), for example, analyzed the general concept of function proposed in 1939 by the Bourbaki group, as "an ordered triple (X, Y, f), where X and Y are sets and f is a subset of $X \times Y$, such that if $(x, y) \in f$ and $(x, y') \in f$, then $y =$

y'" and identified thirteen epistemological obstacles for the comprehension of the concept of function, of which we highlight six for the present study:

1. obstacle linked to a mathematical philosophy that is not interested in practical problems;
2. difficulties in identifying what is changing and which objects vary in a function;
3. not thinking of variables, but rather in terms of equations and unknown values to be extracted from them;
4. taking the order of the variables (dependent or independent) as irrelevant;
5. laws in Physics and functions in Mathematics have nothing in common;
6. the name 'functions' can only be given to describable relations by means of analytic formulas.

In this sense, working with empirical data, using various kinds of graphs in the search for relations that are established among the variables, can help to present and explain the mathematical functions, giving sense to the teaching of such concepts and boosting the learning thereof.

TEACHING FUNCTIONS IN BASIC EDUCATION

Mathematics textbooks used in the 8th grade generally present functions starting from the regularities observed in numerical, geometrical patterns or from the empirical observation of the relation between two variables linked to everyday life, or to natural or social phenomena.

In all of these examples, the relations between the variables are of a deterministic nature and are determined *a priori*. Thus, in a graphic representation of the function, the resulting variable (dependent), denoted by the letter Y (uppercase), is always placed on the ordinate and the generator variable (independent), denoted by the letter X (uppercase), on the abscissa. This way of presenting the functions does not favor the questioning and discussion of the relation that is established between the variables, in terms of what the dependent variable is and what the independent variable is.

Another important question relative to the construction of the graphs concerns the related magnitudes. In general, the examples of mathematical functions limit the values of X to unit values and its graph is constructed with values that pass through the origin (..., -3, -2, -1, 0, 1, 2, ...), therefore, the students rarely come face to face with calibration of scales, since these are already pre-determined.

Some textbooks use statistical graphs to present the concept of functions, very frequently as line graphs that describe phenomena in function of time, known as time series (Figure 1) and, more rarely, the point (or dispersion) diagram (Figure 2). In the case of Figure 1, the authors of the book ask the students to "invent a rule for the function" and construct a table corresponding thereto. In reality, what is being requested to the students is that they estimate the trajectory of the decrease of the illiteracy rate, based on the figure-form to arrive at an algebraic expression that models that trajectory. The authors of Figure 2, on the other hand, present the concept of BMI (body mass index), which is obtained dividing the mass by the square of the height) and request the reading of the BMI for a person that weighs approximately 70 kg. It can be observed that Figure 2 shows the variability of the BMI, since by setting the height at 1.70 m, there is a wide variation for the mass. This is a way to make the connection to the concepts of Statistics, which can help to break down the epistemological obstacles inherent to the concept of function, however the authors do not explore that potential.

Figure 1. Example of a line graph.

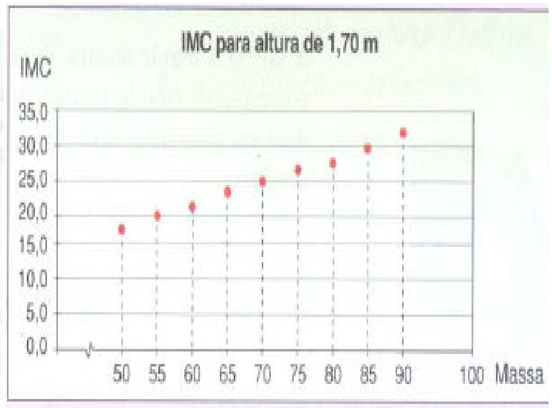

Figure 2. Example of a dispersion diagram.

"STATISTICAL FUNCTION" AND ITS CONTRIBUTION IN TEACHING FUNCTIONS

In Brazil, there is a current trend to utilize interdisciplinary projects in the schools based on a generating and transversal theme (Silva C., 2007). However, what is generally observed is that the projects are summarized to the gathering of data and restricted to qualitative variables and the treatment thereof (counts, percentages, bar graphs and circular graphs); and rarely does the project's guiding question get answered, becoming lost in the course of surveying the data. In this context, rare are the projects that propose works in the paradigm of scientific research; consequently, the data collected and the treatment thereof contribute little to awaken scientific thought and the formation of mathematical concepts.

On the other hand, despite the attempt of contextualization on the part of the textbooks, it can be observed how far away we are from working in the paradigm of the scientific method, in the observation of phenomena that can be modeled by functions. Analyzing math textbooks used in Primary Education, Silva U. (2007) shows an example that attempts to put the function into context in the scope of biological sciences: "The following graph shows the growth of a plant over a period of 10 days. Observe that the height of the plant in centimeters (h) varies according to the time in days (t)." However, that attempt fails when a deterministic linear function (h = 2t) is presented.

In the first place, if a plant is actually picked randomly, the trajectory of its growth would hardly be linear. In the second place, this plant constitutes only one sample of the plants of its species, therefore the risk is run that its growth trajectory does not represent the growth of the species, due to the environmental and genetic variability that influences the growth of each individual plant.

However, the teacher can request each one of the students in the class (n) to conduct the experiment, i.e., plant a seed, observe the growth, measure and record the plant height each day. As an outcome, there will probably be *n* different trajectories, i.e., a sample of *n* plants, being that each trajectory can be modeled by a mathematical function of the type: $h_i = f(t_i)$, where i refers to the plant of the i-th student, as developed by Lehrer and Schauble (2002), whose statistical concept involved is the natural variation inherent to observable phenomena.

If the teacher puts the *n* trajectories on a single graph, indicating only the points observed without uniting them with lines, there would be a dispersion diagram, and would automatically come into conflict with the definition of mathematical function of h in t, since for each specific value of time ($t = t_0$), up to *n* distinct values may be encountered, not obeying one of the pre-requisites of the concept of mathematical function.

Actually, the teacher is no longer faced with a function of height and time, but is now faced with a "statistical function", i.e., in addition to the relation between height and time, for each time ($t = t_0$) there exists a distribution of empirical frequency of the height of the plants for that age, from that sample (Figure 3). The distribution of theoretical frequency of the height of the plants of that species, once the age is set, can be modeled by a mathematical function, albeit of a probabilistic nature (Figure 4) due to the natural variation of the plant height. Furthermore, the teacher is faced with the measurements of a random sample of *n* plants; so if the experiment was replicated by another group of students, the results will not necessarily be equal.

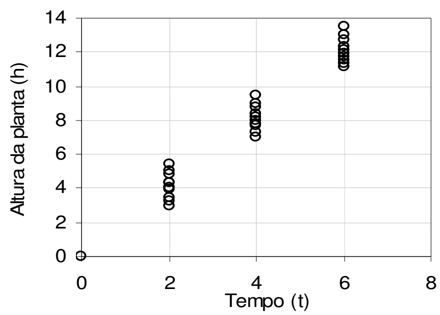

Figure 3. Example of the height of 10 plants

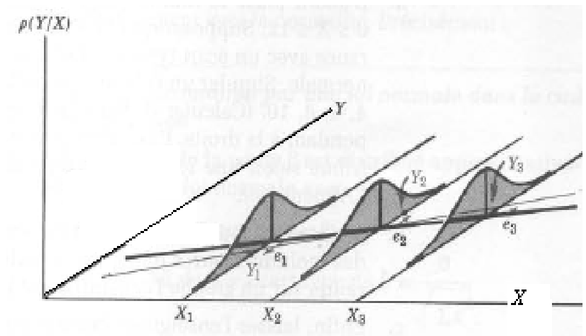

Figure 4. Conditional distribution of Y given a fixed X

This process demonstrates the need to have worked the continuous variables in an unvarying manner, i.e., besides working the concepts of median and standard deviation and the graphic representation thereof in histograms, it would be interesting to present the diagram of unvarying points, those of the branch and leaves, and even the boxplot, which are intuitive concepts but that are not contemplated in Brazil's National Curricular Parameters. But, how we can take advantage of this kind of "statistical function" and its graphs during the teaching of mathematical functions in Primary School without complicating the concept of function?

The work developed by Silva E. (2007) shows that it is possible to explore the elementary concepts of Statistics from a perspective of scientific research, in the learning of functions. The author elaborated a teaching intervention based on Leonardo da Vinci's "Vitruvian Man" to analyze with his students the plausibility of the "perfect" relation between a person's arm span (Y) and height (X), formulated by da Vinci. If this hypothesis were true, the relation could be modeled by a linear identity function ($Y = X$), i.e., the one that passes through the origin, with an angular coefficient equal to 1. In order to contextualize this theme, the students visited the exposition "Leonardo da Vinci—The Exhibition of a Genius," which was currently showing in their city. Afterwards, in the classroom, the students measured their own arm span, height, weight, among other variables; and finally treated the data and constructed graph with pencil and paper as well as on a computer. This experience led the students to make the decision on statistical concepts related to the concept of function.

First, the students must decide which relations were established among the variables involved, i.e., identify whether there was any relation of dependence between the two variables: is height a function of arm span, or is arm span a function of height? And what about weight and height? A second challenge that the student faced was the existence of different magnitudes in the variables observed. The weight, measured in kilos, varied from 40 to 90, and the height and arm span, measures in centimeters, varied from 150 to 190. Once the relation of dependence between the variables was decided, the students proceeded to the representation thereof on the Cartesian plane, giving meaning to the representation of the dependent variable on the ordinate and the independent variable on the abscissa.

The drawing of the graph implied the calibration of the scale, i.e., the need to establish the proportionality between one square of the graph paper and its meaning in terms of centimeters and kilos. In this example, the students saw themselves represented on each point of the dispersion diagram. Figures 5 and 6 show the data drawn with EXCEL.

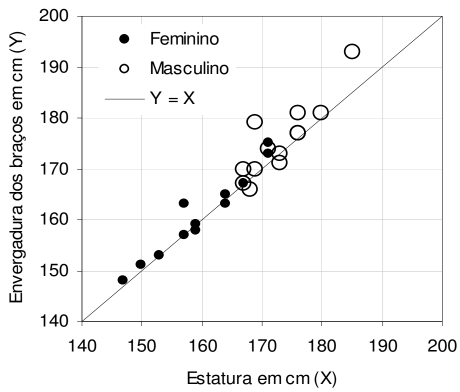

Figure 5. Relation between arm span and height.

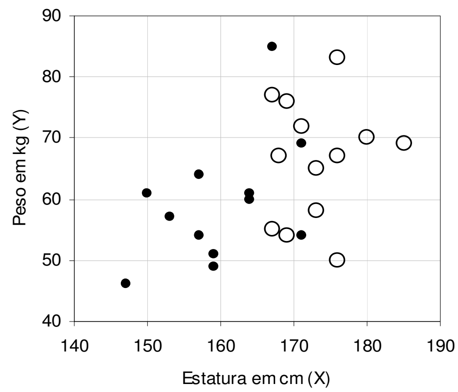

Figure 6. Relation between weight and height.

It was automatically observed that there were students who had the same height but whose arm spans were not the same, bringing about the discussion of natural variation inherent in most phenomena observed, as well as possible errors in measurement.

If the relation between the height (X) and the arm span (Y) was perfect, all of the points would be on the straight line of the equation Y = X, which did not happen (as can be observed in Figure 5). The cloud of points reveals the nature of the relation between the two variables: linear and direct, i.e., when one grows the other one grows and vice versa. However, this does not happen in such a clear way with the relation between weight and height in Figure 6, which shows a widely scattered cloud of points, due to the greater variability in weight, which makes the modeling more complex.

A third challenge proposed to the students was the modeling of the equation of the straight line, uniting any two points. The students realized that there can exist many straight lines and that this was easier in the case of arm span and height than in the case of weight and height. In that case, the resulting question was: how to choose a straight line? Which one best describes the relation among the variables? And what if measurements were taken of another group of students, will the cloud of points be the same? So, what is the true relation? What is this relation by gender? Vis-à-vis these data, can the hypothesis of the perfect relation be accepted or should it be refuted? These challenges can certainly, if worked properly, contribute significantly to the formation of the concept of function.

FINAL CONSIDERATIONS

This article sought to show the importance of the elementary concepts of Statistics in the formation of the concept of mathematical function, from a perspective of the intradisciplinary and interdisciplinary nature thereof, having as a backdrop the logic of the method of scientific research. It is believed that working in a contextualized manner, where the students traverse all of the phases of the scientific method, collecting empirical data, analyzing and modeling such data, can contribute toward the mathematical formalization of the functions and the development of the scientific and critical spirit of the students, developing statistical competence in the sense defined by Rumsey (2002) and as advocated by the National Curricular Parameters (Brazil, 2002, p. 42):

Therefore, the teaching of Mathematics is responsible for guaranteeing that the student acquire a certain flexibility in dealing with the concept of function in diverse situations and, in this sense, through a range of problem-situations of mathematics and of other areas, the student can be encouraged to seek a solution, adjusting his or her knowledge about functions in order to build a model for interpretation and investigation in Mathematics.

The use of the collection and analysis of data in the teaching of functions can contribute to the elimination or minimization of the epistemological obstacles identified by Sierpinska (1992); namely, the abstract model presents a relation with the observed reality and justifies it, which makes the definition of the dependent/independent variable an indispensable task and puts it into context for the students. It is possible to observe that for any one reality, there can exist different algebraic expressions for the phenomenon observed, minimizing the “power” of the formal operations with algebraic expressions.

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