Games of interpretation, anticipating thought and coordination between verbal and algebraic register: key-aspects in the analysis of students' proofs in elementary number theory[1]

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Abstract

This work is part of a wide-ranging long-term project aimed at fostering students' acquisition of symbol sense through teaching experiments on proof in elementary number theory (ENT). Our aim is to analyze the use and the role of algebraic language in the development of such proofs. In this paper we present the analysis and classification of students' behaviour in facing the proof of a conjecture while working in small groups. The analysis of students' protocols was made by reference to the following interpretative-keys: the application of specific conceptual frames, the games of interpretation between different frames, anticipatory thoughts, the use of conversions and treatments and coordination between different registers of representation. Our analysis highlights the incidence of anticipatory thoughts and of the flexibility in the coordination between different frames and different registers of representation in the development of proof in ENT. Moreover our work testifies the effectiveness of the analysis of students' discussions during small group activities as a methodological instrument to highlight all these aspects.

1. Introduction

Many research studies support an approach to algebraic language related to the development of reasoning. Arcavi (1994, 2005), for example, claims that, in addition to stimulating students' abilities in the manipulation of algebraic expressions, teachers should make them see the value of algebra as an instrument for understanding, expressing and communicating generalizations, the establishment of connections, or the production of argumentation and proof. A central aspect in Arcavi's approach to algebraic language is the concept of symbol sense. The author chooses to characterize symbol sense highlighting, through meaningful examples, the attitudes to stimulate in students to promote an appropriate vision of algebra. Particular attitudes that he names include: the ability to know when to use symbols in the process of finding a solution to a problem and, conversely, when to abandon the use of symbols and to use alternative (better) tools; the ability to see symbols as sense holders (in particular to regard equivalent symbolic expressions not as mere results, but as possible sources of new meanings); the ability to appreciate the elegance, the conciseness, the communicability and the power of symbols to represent and prove relationships. For these reasons, Arcavi argues that students should be introduced to algebraic symbolism from the beginning of their studies through specific activities that encourage in them an appreciation of the value and power of symbols.

Many researchers share a similar vision of the approach to the teaching of algebra. Among them, Bell (1996), states, in particular, that it is necessary to favour the use of algebraic language as a tool for representing relationships, and to explore aspects of these relationships by developing those manipulative abilities that could help in the transformation of symbolic expressions into different

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forms. This idea is strictly connected with Bell's description of "the essential algebraic cycle" as an alternation of three main typologies of algebraic activity: representing, manipulating and interpreting. Similar observations are also found in Wheeler (1996), who asserts the importance of ensuring that students acquire the fundamental awareness that algebraic tools "open the way" to the discovery and (sometimes) creation of new objects. Kieran (2004) also stresses the importance of devoting much more time to those activities for which algebra is used as a tool but which are not exclusively to algebra (global/meta-level activities according to Kieran's distinctions). In fact, these activities help students developing transformational skills in a natural way since meaning supports manipulations (Brown 2004).

Proof is certainly one of the main activities through which helping students develop a mature conception of algebra. We adopt Wheeler's idea that activities of proof construction through algebraic language could constitute "a counterbalance to all the automating and routinizing that tends to dominate the scene". Selden and Selden (2002) suggest that ENT is "ideal for introducing students to reasoning and proof" because it makes students deal with familiar objects and reduces the level of abstraction required. We believe that activities of proof in ENT would both provide students with the opportunities they need to progress gradually from argumentation to proof and help them to appreciate the value of algebraic language as a tool for the codification and solving of situations that are difficult to manage through natural language only (Malara 2002).

We agree with Zazkis and Campbell (2006), who state that "the idea of introducing learners to a formal proof via number theoretical statements awaits implementation and the pros and cons of such implementation await detailed investigations" (p.10). In order both to investigate these aspects and to foster the diffusion of activities of proof in ENT in school mathematics, we are working with pre and in-service teachers (Cusi & Malara 2007) and with upper secondary school students. The activities we focus on in this paper refer to students and are part of a long-term experimentation realized in some classes (10^{th} grade) of a *Liceo Socio-Psico-Pedagogico*.

1.1 The choice of the research context

Liceo Socio-Psico-Pedagogico is an upper secondary school originally aimed at educating future primary school teachers. In this kind of school Mathematics has a minor role. In fact it is taught only 4 hours per week and the curriculum is more limited than that at schools with scientific or technical courses. An important consequence is that many of the students attending this school are neither interested, nor inclined for mathematics. The classes involved had already had a traditional approach to algebra, during the previous school year. We chose to propose them this teaching sequence, in order to highlight problematic aspects related to possible conflicts between an existing view of algebraic language and the new awareness the activities are meant to stimulate. Moreover, due to the difficulties at syntactic level, many of these students had shown at the beginning of the school year, we thought that proposing them this type of activities would have been a good research context to verify whether it is possible, though an approach like this, to facilitate little steps towards a vision of algebra as a tool for generalising, reasoning and proving.

1.2 Methodology of work with students

Aiming at making student acquire an effective symbol sense, we planned and experimented a path for the introduction of proofs in ENT through different gradual phases of work. The path can be subdivided in 6 main phases, characterized by the following activities: (1) Translations from verbal to algebraic language and vice-versa; (2) Study of the relationship between properties of a given formula and properties of the variables it contains; (3) Analysis of the truthfulness/falseness of statements concerning natural numbers and justification of the given answers; (4) Exploration of numerical situations, formulation of conjectures and related proofs; (5) Analysis of proofs; (6) Construction of proofs of given theorems. The path (about 20 hours) was articulated through small-groups activities (8 groups were audio-recorded), followed by collective discussions (audio-recorded) on the results of the small-group activities. In this work we will dwell on a central moment in the path: the small-groups' work aimed at constructing the proof of some conjectures they produced starting from numerical explorations. In particular, we will present the main results of our analysis of group discussions when students were trying to prove one of the conjectures.

2. Theoretical framework which support our analysis of students' discussions

Many different competencies are required of a student who has to face proof problems in ENT. In particular, he/she has to: (a) know the meaning of the mathematical terms in the problem text and interpret them correctly by reference to it; (b) translate correctly from the verbal to the algebraic language; (c) be able to interpret the results of the transformations operated on the algebraic expressions in relation to the examined situation; and (d) control the consequences of his/her assumptions.

We identified a set of theoretical references that are both appropriate to the analysis of the transcripts of group discussions dealing with proofs, and in tune with the view of algebra that we are promoting. The main reference in our research is the work by Arzarello *et Al.* (1994, 2001). The authors propose a model for teaching algebra as a *game of interpretation* and highlight the need for the promotion of algebra as an efficient *tool for thinking*. An awareness of the power of the algebraic language can be developed only once the student has mastered the handling of some key-aspects that arise in the development of algebraic reasoning. In particular, the authors highlight the use of *conceptual frames* defined as an "organized set of notions, which suggests how to reason, manipulate formulas, anticipate results while coping with a problem", and *changes from a frame to another* and from a knowledge domain to another as fundamental steps in the activation of the interpretative processes. According to the authors, a good command in symbolic manipulation is related to the quality and the quantity of anticipating thoughts which the subject is able to carry out in relation to the effects produced by a certain syntactic transformation on the initial form of the expression. Boero (2001) also argues that *anticipation* is a key-element in producing thought through processes of transformation. The author defines anticipating as "imagining the consequences of some choices operated on algebraic expressions and/or on the variables, and/or through the formalization process". In order to operate an efficient transformation, the subject needs to be able to foresee some aspects of the final shape of the object to be transformed in relation to the target. Arzarello *et Al.* stress that the ability to produce anticipations strictly depends on changes in the frame considered in order to interpret the shape of the expression.

Another theoretical reference that we take as fundamental for analysing students' management of meaning in algebra is the concept of *representation register* proposed by Duval (2006). The author defines representation registers those semiotic systems "that permit a transformation of representations". Among them, he includes both natural and algebraic language. Duval asserts that a critical aspect in the development of learning in mathematics is the ability to change from one representation register to another because such a change both allows for the modification of transformations that can be applied to the object's representation, and makes other properties of the object more explicit. According to the author, real comprehension in mathematics occurs only through the coordination of at least two different representation registers. He analyzes the functions performed by different possible typologies of transformations. He distinguishes between *treatments* ("transformations of representations that happen within the same register") and *conversions* ("transformations of representation that consist of changing a register without changing the objects being denoted"), highlighting both the fundamental role of each of these typologies of transformations and the intertwining between them.

3. Research hypothesis and aims

Our hypothesis is that the production of good proofs in ENT depends upon the management of three main components: (a) the appropriate application of frames and coordination between different frames; (b) the application of appropriate anticipating thoughts; and (c) the coordination between algebraic and verbal registers (on both translational and interpretative levels). Our aim is to investigate the following aspects: (1) Effectiveness of the theoretical references we selected as tools for analyzing and classifying students discussion about proofs in elementary number theory; (2) Identification of the essential components for good productions in this context; and (3) The role played, by the three components we singled out (application and coordination between frames, anticipating thoughts, coordination between verbal and algebraic register) and the mutual relationships between them. In this work we will present a sample of prototype-productions helpful to verify our hypothesis and to highlight that the lack or unsuccessfully application of one of these components leads to failure and/or blocks of various types.

4. Research Methodology

Theoretical models we used helped us to identify some interpretative keys for both the analysis of protocols and their subsequent classification. Our analysis focused on the following: (1) The conceptual frames chosen to interpret and transform algebraic expressions and the coordination between the different frames appropriate to those same expressions; (2) The application of anticipating thoughts; and (3) The conversions and treatments applied and the coordination between verbal and algebraic registers.

4.1 Why small-groups activities?

The study of small-groups work is proposed by researchers with different aims: some of them are interested in the effects of these kind of activities as instruments to promote learning (Barnes, 2005), other researchers focus on the dynamics which characterize the students' mathematical discourses while they are working in groups (Ryve 2006), others aim at highlighting how individuals re-construct mathematical concepts through small-groups interaction (Vidakovic & Martin, 2004). Our choice of making students work in small groups is, instead, motivated by a different reason. Our conviction is that only when students are involved in a communication it is really possible for us to produce an in-depth analysis of the coordination between verbal and algebraic register in the construction of proofs in ENT. We believe that the analysis of the sole written protocols is not enough to highlight students' actual interpretations of algebraic expressions they construct. The need to communicate their reasoning to others forces students not only to verbally make what they are writing explicit, but also to explain both the objectives of the transformations they carry out and their interpretation of results.

5. The problem and its *a priori* analysis

The problem, on which we focus in this paper, is the following: *"Write down a two digit number. Write down the number that you get when you invert the digits. Write down the difference between the two numbers (the greater minus the lesser). Repeat this procedure with other two digit numbers. What kind of regularity can you observe? Try to prove what you state"*.

The regularity to be observed is that the difference between the two numbers is always a multiple of 9; precisely it is the product between 9 and the difference between the digits of the chosen number. The proof requires the polynomial representation of each number: since a number of two digits m and n can be written as $10m+n$, where $m>n$, the difference can be represented as $10m+n-(10n+m)$. Through simple syntactical transformations it is possible to turn the initial expression into a form that makes the required property explicit: $10m+n-(10n+m)=9m-9n=9(m-n)$.

The initial conceptual frames to which the statement of the problem refers are ‘difference between numbers’ and ‘two digits numbers’. It can be assumed, therefore, that the student will not automatically choose the ‘polynomial notation’ frame to represent the problem and apply the necessary simple treatments to make the conjectured property explicit (some students might apply the ‘positional representation of a number’ frame and then get stuck). The reference to the ‘divisibility’ frame, which allows them to foresee the desired final shape of the expression after correct treatments (i.e. $9 \cdot k$, where k is a natural number), is, instead, less problematic. Possible blocks in the treatments to perform on the initially constructed polynomial expression can be ascribed to interpretative difficulties, which are strictly related to students' inability to correctly anticipate the final shape of the considered expression (it is necessary to recognize the transformation that leads to an expression that can be easily interpreted in the final frame ‘divisibility’). Finally, we make some observations about possible student behaviour. Many students could end their numerical explorations after having observed that the difference between the two numbers is always a multiple of 9, without recognizing the relationship that exists between the two digits of the first number and the difference between the two numbers. Consequently, the analysis of the final expression could provide another index of a students' interpretative abilities, in that access to the new meanings it embodies depends on those abilities.

6. The analysis of prototype-productions

In the analysis of small group work, we singled out the incidence and the interrelation between the following: (a) the activation of and coordination between frames, (b) ability in the game of interpretation required to produce the proof, (c) display of appropriate anticipating thoughts, (d) ability to correctly perform treatments and conversions, and (e) ability to coordinate verbal and algebraic registers. Through our analysis we were able to highlight four main categories of prototype-productions: (C1) Partial reference to algebraic language and presence of blocks; (C2) Application of a suitable frame, but inadequate conversion and incorrect interpretation of the produced expressions; (C3) Appropriate activation of frame and conversion not supported by semantic control and anticipating thoughts; (C4) Good coordination between frames and good interpretation of the expressions in the applied frames. In this paragraph we will present three examples of prototype-protocols: the first belongs to C2, the second to C3, the third from C4. We chose not to propose examples from C1 because they are very poor productions, characterized by the reference to numerical examples only and by attempts of formalizing the conjectured property not aimed at the proof of the same conjecture. The examples referred to C2 and C3 were chosen because they highlight how students’ interaction allows to identify the reasons of erroneous conversions and the difficulties in the interpretation of expressions. The example referred to C4, instead, better dispels what kind of behaviours turn out to be most efficient for the production of a proof in this context.

6.1 Example referred to C2: Application of a suitable frame, but inadequate conversion and incorrect interpretation of the produced expressions

After having considered many numerical examples, students A, C and N conclude that the considered difference is always a multiple of 9. The following dialog represents the proving phase. C (27) : Let us do with letters. N (28) : It is more complicated. C (29) : It will be *10x* … plus … A (30) : …plus *y (they write 10x+y)*[2] C (31) : If we invert the digits, it will be *y+10x* N (32) : Yes. A (33) : and now … we have to do the difference C (34) : (*she writes and reads*) *10x+y* A (35) : Let us put it in brackets C (36) : minus ... (*y+10x*) C (37) : it becomes *10x+y-y-10x* N (38) : I think there is a mistake because the result is zero … A (39) : It become *10x+10x*? N (40) : No, they cancel each other out. C (41) : Meanwhile, let us write: (*she dictates*) they are all multiple of 9 … It is not simple … N (42) : We are not able to prove it. It is difficult. C (43) : We have 10x+y and it represents the number … Then we have to … A (44) : *(she reads)* 'when you invert the digits' … C (45) : It is the same of having 1 and … C (46) : It is as if we take it on this side, so *y* should be take on the other side. C (47) : however, if we take 10 on this side, it will be left a … A-N-C (48) : one! C (49) : So it is not 10x. I think it is x … A (50) : Let us try! C (51) : So it would become *10x+y-(y+x)*. The two *y* cancel each other out, so they will be left *10x-x*. Exactly *9x*! We were able to prove it! C (52) : (*C. is looking to the numerical examples*) But here I can see something more, I think. I can see that, practically, this is … Look what I noticed (*she is looking at the differences 86-68, 92-29, 76-67, 52-25*) … if you subtract the two tens, 8-6, you have only to consider the product between 9 and the difference between the two tens: 9 times 2 is 18; 7-6 is 1, 9 times 1 is 9; 5-2 is 3, 9 times 3 is 27. A (53) : We have to write it down. I would have never noticed it! C (54) : (*she dictates*) It is always a multiple of 9 and we can observe that the result of the subtraction … you have to subtract the two tens and to multiply the result by 9. C (55) : Do you know how I though of it? Because I saw 9x and I said "it is a multiple" because there is 9 times x. Then I said "but … what is x? x is the tens!". Then I tried to do x minus x. A+N (56): Good!

[2] The difficulties we hypothesised in the identification of the initial frame are not highlighted by this protocol because students have faced the problem of the representation of two and three-digit numbers in a previous activity.

This protocol can be subdivide in three key-moments: (1) *Initial conversion and first treatments* (lines 27-37); (2) *Identification of a problem, modification of the conversion and new treatments* (lines 38-51); (3) *Attempt of interpretation of the obtained expression and refinement of the conjecture* (lines 52-56). The protocol highlights, first of all, the central problem in groups' works on this proof, that is the difficulties students met in representing the number obtained inverting the two digits. Success in this conversion requires both a good coordination between the 'positional notation' and 'polynomial notation' frames and a complete internalization of the last. Initially C. carries out a *first erroneous conversion* (line 31), translating this concept through the expression $y+10x$ (she only changes the order of the addends). The students correctly interpret the natural language term "invert" when they work on numerical examples in order to formulate the conjecture. Afterwards, however, when they have to carry out a conversion into algebraic register, the concept "exchanging the place" is translated through the pure exchange of the order of the monomials which constitute the polynomial $10x+y$. The difference (zero) they obtain starting from this erroneous conversion lead them to detect the inaccuracy of their initial conversion and to look for a new correct one. They detect a mistake in having supposed that $10x$ should represent the units digit (only because it is on the right side of the polynomial $y+10x$). So they decide to correct this mistake, substituting x instead of $10x$, but they do not consequently modify the representation of y as tens-digit. Therefore, writing the polynomial as $y+x$, they carry out again an incorrect conversion. Probably because of the prevailing of the anticipating thought they carry out (expecting a multiple of 9, they only concentrate on the factor 9 when they look at the expression $9x$), once they obtain $9x$ as the difference between the two numbers, they do not immediately subject the new result to a careful interpretation. Only afterwards C. interpret x as the tens-digit of the initial number and decide to investigate the considered examples in order to refine their conjecture. C. concentrates on the tens-digits of the two numbers (x and y in the correct representation) and observes, starting from examples, that the result is obtained multiplying 9 by the difference between those digits. This observation, however, does not help her in critically interpreting the expression $9x$. In her final intervention, she even tries to translate into algebraic language, through the expression $x-x$, the difference between the two tens, but she is not able to 'grasp' the gap between the algebraic representation she proposes and her verbal considerations.

6.2 Example referred to C3: Appropriate application of frame and conversion not supported by semantic control and anticipating thoughts

In the following dialog the three students G, B and A, after having read the text of the problem, decide how to organize the phase of work.

G (1): You can try to work with the general case (*she refers to B*), while we will try to work with the numerical case.

B (2): Ok. But … in the sentence, how is 10y+x?

A (3): 10x+y!

G (4): I think that the greater number is always 10y because it is multiplied by 10.
A (5): But she has to decide (*which is the greater*) because she has to work with letters.
B (6): But, in the second number the units digits becomes greater … this one (*she indicates y*)
G (7): Ah, it's true!
A (8): I think that there is 10 only to show the tens digit!
B (9): But I don't know what digit is the greater, because I'm considering a general case.
A (10): Bah… you can decide it! Let's take x>y!
B (11): Ok. Let's work ...

A and G start to confront each other to clarify what are the requests of the problem and what kind of numerical examples can be considered to formulate the conjecture. After a brief discussion they conclude that they can refer to every two-digits number. While A and G keep on with the analysis of numerical examples, B works silently. She is able to perform the correct conversion, representing the considered difference with 10x+y-(10y+x). Afterwards she performs correct treatments on this expression, obtaining 9(x-y), and she decides to illustrate her result to A and G.

B (19): I obtained this thing … Why 9? 9 is 9! 9 is odd! Is it possible that the result is always an odd number?
A (20): No. Consider 20! The difference is 18!
G (21): I sincerely can't find a regularity …
B (22): I could only find that the result is 9 multiplied by x-y, but … why does 9 is here? There is 9 only because there is 10!
G (23): Let's try with 28 … 82-28 … the result is 54! So … What have these numbers in common???
B (24): I found it!! I found it!! If I choose 65 and 56, the difference is 9. In the algebraic case the result is 9 multiplied by (x-y)!
G (25): Please, explain it!
B (26): Because, independently from the initial number, the difference is always 9.
G (27): No! Consider 82 and 28!
B (28): What a pity! I liked this observation!
A (29): Consider 14 and 41
B (30): Wait a moment … here (*she refers to the examples she chose*) we pass from a ten to the next ten. I found it! Only if we start from a number whose digits are consecutive, the difference is 9!!! 34 and 43 … All the numbers have consecutive digits!
G (31): It is true! 54 e 45!
B (32): 12, 23, … Do you understand? 1 and 2 are consecutive numbers.
A (33): 14 and 41? 15 and 51?
B (34): No! The two digits must be consecutive! When they are consecutive, the difference is always 9!
A (35): So … what does it happen?
B (36): I don't know … It happens that the difference between the numbers is 9. If you look at the algebraic case … Can you see that it is always 9 multiplied by something?
A (37): Only if the digits are consecutive the difference is 9?
G (38): It is strange …
B (39): I don't know why …
G (40): But … I think that the distance between the numbers is not the only reason …
(*silence*)
B (41): … It is always a multiple of 9!!!
A (42): In what sense?
B (43): Let's try! 52-25! The result is 27!
A (44): Also if we choose 15 and 51 …the result is 36!
B (45): They are all multiple of 9! Can you see that every case is the same! Tell me other numerical examples!
A (47): 51-15 is 36
G (48): 52-25 is 27
B (49) : 21-12 is 9, which is a multiple of 9!
G (50): So we can observe that the result is always a multiple of 9.

This excerpt could be subdivided in three key-moments: (1) Organization of the working group activity and discussion about the interpretation of the problem's text (lines 1-11); (2) Attempt to interpret the expression produced during an 'algebraic exploration' of the problem situation (lines 19-40); and (3) Formulation of the conjecture (lines 41-50).

The first phase of this working group excerpt is devoted to the organization of the activity through a subdivision of the different roles to be assumed by the three students. The three students also confront each other to the correct interpretation of the problem's request. G's observation about what is the greater number (line 4) reveals that the student has not interiorized the meaning of the polynomial representation of a number and that she does not coordinate the polynomial notation and the positional notation frames. A and B, instead, dispel to be aware about the connection between the limitations to be imposed on the digits of the chosen number and the relation between the two numbers (lines 8, 9, 10).

The choice to proceed separately (two students work on numerical examples only and the third works on the algebraic formalization of the considered difference) turns out to be not effective. In fact, while the analysis of numerical examples does not help A and G in formulating a conjecture, B correctly performs both the conversion from natural to algebraic language through the activation of the correct frame (polynomial notation) and the following treatments on the considered expression. However, the total absence of anticipatory thoughts about the objective of the algebraic manipulations she operates, blocks B's interpretation of the obtained expression $9(x-y)$. B's first interpretation of the expression as the representation of an odd number (line 20) is immediately refuted by a counterexample proposed by A (line 21). At this point, B decides to refer to numerical examples in order to meaningfully look at the obtained expression. The choice of the numerical examples she considers (only numbers whose digits are consecutive) suggest her that the difference is always 9 (line 24). Now the presence of an anticipatory thought (the difference is 9) negatively influences B's interpretation of the expression $9(x-y)$. Again the other students propose a counterexample against B's conjecture (lines 27 and 29).

At this point, B does not try to re-interpret the expression and limits herself to look at numerical examples to understand what are the conditions under which the regularity she first observed (the difference is 9) is valid (lines 30 and 34). Although her correct observation about the interrelation between the digits of the initial number and the difference between the two numbers, again B is not able to correctly re-interpret the expression $9(x-y)$, focusing on the role assumed by the factor $(x-y)$ (lines 36 and 39).

The activation of the frame 'being multiple of' and B's final correct interpretation of the expression as a multiple of 9 (line 41) is the result of her analysis of numerical examples only. B's troubled conquest of an only partial interpretation of the expression $9(x-y)$ and her necessity to refer to numerical examples to understand what she obtained testify that if algebraic manipulations are not guided by an objective significant interpretations are blocked. In other words, *blind manipulations* easily lead to failures. An evidence of this problematical aspect is the fact that, paradoxically, the working group activity ends with the formulation of the conjecture.

6.3 Example from C4: Good coordination between frames and good interpretation of the expressions in the applied frames

Z (1) : (*she reads the text of the problem*) 'Write down a two digit number'.
C (2) : … 'and the number that you get when you invert the digits'.
Z (3) : 52 minus 25. Am I right?!
C (4) : Why (*do we consider*) 52 minus 25 and not 25 minus 52?
Z (5) : Because it can become negative (*she refers to the considered difference*).
G (6) : 'Write down a two digit number' … so 10x+y.
Z (7): No, we have to consider natural numbers.
C (8): We have to start with couples of numbers, like 21 minus 12. 12 because it is its inverse. And the result is 9. Then we have to take other numbers: 53-35, that is 18. They are all multiple of 9. Let's try with other numbers: let's try with 25: 52-25.
Z (11) : It's 27.
C (12) : Ok. Let's consider the algebraic example: 10x+y-(10y+x). But we don't know which is the greater number and which is the smaller.
Z (13) : It's the same. When we finish we only have to change the signs! Let's do the difference.
C (14) : The result is: 10x+y-10y-x. And it is … 9x ... Ok ... minus 9y. (*They write 9x-9y*)
Z (15) : Then we can highlight the factor 9.
C (16) : 9(x-y)
Z (17) : Ok, we have proved it.
G (18) : What is x-y?
C (19) : Wait a minute… x-y is the difference … between x and y. They are the digits. Therefore the result is 9 multiplied by the difference between the digits. In a numerical case, for example 52-25, we obtain 9 multiplied by 5-2.
Z (20) : Oh, that's nice!
C (21) : Let's write it …

This excerpt can be subdivided in three fundamental moments: (1) An initial brief *discussion devoted to the formulation of the conjecture* (lines 1-11); (2) the proof of the conjecture (lines 12-17); and (3) the analysis of the obtained algebraic expression and subsequent refinement of the conjecture (lines 18-21).

The protagonist of this working-group activity is essentially C. The student dispels confidence not only when she analyzes numerical examples, but especially when she proposes an algebraic approach to the proof of the conjecture. After a brief moment devoted to numerical explorations, C concludes that the considered differences are always multiple of 9. A significant aspect is the use of the term '*algebraic example*' by C to refer to the algebraic expression which represent the formalization of the considered difference (line 12). The introduction of this term, in fact, testifies that C is aware that this algebraic expression represents a generalization of every numerical example to be considered and that, in a certain sense, this approach is the only way to prove their conjecture. Afterwards, without hesitation, C operates a good conversion from natural language to algebraic language, correctly formalizing the considered difference (line 12). C's doubt about the correctness of the expression she produced (she asks how to know what number is the greater one), testifies that the student is trying to control the relevance of her conversion in relation to the verbal description proposed in the text of the problem. Although proposing some limitations for the variables x and y constitutes the best approach to this proof, Z shows to have understood the problem raised by C and dispels a good anticipatory thought when she answers that the property of the produced expression they want to highlight (it is a multiple of 9) do not depend on the sing of the same expression (line 13). Z's anticipatory thoughts and reference to the correct final frame (the frame 'being multiple of') are also testified by the suggestion she gives to C about the treatments to be operated on the expression *9x-9y* (line 15).

G's question (line 18) constitutes a final stimulus for C to further interpret the expression *9(x-y)* through an effective coordination between the 'polynomial notation' frame and the 'being multiple of' frame. In this way C is able to propose a refinement of the initial conjecture.

7. Conclusions

Our analysis of students' discussions during small group activities turned out to be an effective methodological instrument to highlight the key-components for the analysis of proof productions in ENT, that is: (a) The conceptual frames chosen to interpret and transform algebraic expressions and the coordination between the different frames appropriate to those same expressions; (b) The application of anticipating thoughts; and (c) The conversions and treatments applied and the coordination between verbal and algebraic registers.

Moreover our analysis allows us to offer some conclusions with respect to the aspects we wanted to investigate, namely: (1) Effectiveness of the theoretical references we selected as tools for analyzing and classifying students discussion about proofs in elementary number theory; (2) Identification of the essential components for good productions in this context; and (3) The role played, by the three components we singled out and the mutual relationships between them.

The first and second protocols we analyzed helped us in the investigation of the third aspect.

The first protocol highlight the strict correlation between lack of flexibility in coordinating different frames, difficulties in carrying out conversions from verbal to algebraic register and lack of interpretative games in the analysis of the expressions produced. Moreover, it testifies how such correlation causes failures in the production of proofs in ENT. In fact, the three students display rigidity in their use of frames and an incapability of simultaneously manage different frames. Such rigidity make them produce partial or incomplete interpretations of the constructed expressions, so they are not alerted about the non-acceptability of their proof.

The second protocol testifies the strict interrelation between the absence of anticipatory thoughts and failures in the activation of frames and consequently in the subsequent interpretations of the produced expressions.

The rigidities highlighted in the analyzed protocols are shared by other protocols, to which different problems could be add. Summarizing, these are the main problematical aspects we singled out in through our analysis: (a) blocks related to the activation of an incorrect initial frame of reference; (b) blocks in the treatments and in the interpretative processes due to an inability to foresee the expression to be attained by the activation of the correct final frame; (c) difficulties in the choice of the treatments to be operated caused by the absence of anticipatory thoughts; (d) interrelation between lack of flexibility in coordinating different frames and difficulties in carrying out conversions and in interpreting the produced expressions; and (e) interrelation between *blind manipulations* (i.e. produced without a conjecture, therefore without an objective) and blocks in the interpretative processes.

The analysis and the classification we made also allow us to verify the effectiveness of the theoretical elements we selected as tools for analyzing small-groups' discussions for the construction of proofs in ENT.

Finally, the protocol we proposed as an example of category C4 represents an evidence of the fact that an appropriate application and combination of the three components we highlighted (appropriate application of frames and appropriate coordination between them; appropriate anticipating thoughts; appropriate coordination between algebraic and verbal registers) is a necessary condition for the proper development of a proof in ENT.

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