Exploring Connections Between Numeracy and Algebraic Thinking: Some Current Directions in New Zealand

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Abstract

At present there are initiatives to extend the New Zealand Numeracy Development Projects from number into algebra. This paper describes an approach to linking numeracy with students' strategies for solving linear equations. Preliminary data from diagnostic interviews with 500 Year 7 to Year 10 students suggests that there is a hierarchy of sophistication of strategies. The most sophisticated strategy that a student is able to use is associated with the stage of numeracy of the student.

Introduction

A new curriculum for all subject areas in New Zealand schools was launched on 6 November 2007 (Ministry of Education, 2007). The learning area of Mathematics and Statistics is now divided into three strands rather than the previous six, with Number and Algebra being one strand. The implementation of the curriculum in number up to Year 10 of schooling is guided by the Numeracy Development Projects, which provide a framework for children's development in number. The achievement objectives of the new curriculum are grouped to reflect the structure of the Number Framework (Ministry of Education, 2003b), which details the number strategies that students use and the number knowledge required for these strategies. At the lower levels of the new curriculum the Number and Algebra achievement objectives are divided up into *number strategies, number knowledge, equations and expressions,* and *patterns and relationships*. The integration of number and algebra into one strand follows debate within the mathematics education community in New Zealand and within international research (see for example Carraher & Schiemann (2007), Kieran (1992) and Lee (2001)) as to what constitutes algebraic thinking.

The Numeracy Development Projects have been very successful at raising the achievement of New Zealand children in the strand of Number (Thomas & Tagg, 2007; Young-Loveridge, 2007) and various initiatives are currently underway to extend the projects into early algebra. Irwin & Britt (2007) have examined the impact of the Numeracy Development Projects on students' ability to generalise and suggested that students who can access a range of strategies to solve numerical problems are using thinking that is essentially algebraic in nature. These students are the ones who are likely to be able to work with algebraic symbols to express generality. At the University of Otago College of Education we are examining students' strategies for solving linear equations and the relationship of these strategies to numeracy. It is this work in progress that is reported in this paper.

Background

Many students struggle with introductory algebra and teachers have little to guide them in assisting students to learn this important component of mathematics. Little is known about the effect of students' numeracy on the learning of early algebra, or about the strategies that students use to solve equations. There is widespread agreement that algebra is not easily understood by many students. The Cockroft Report in the UK highlighted the fact that algebra is a source of considerable confusion and negative attitudes among pupils (Cockcroft, 1982), while the title of Brekke's (2001) paper, "*School algebra: Primarily manipulations of empty symbols on a piece of paper?"* sums up the situation for many students.

Because much arithmetic in schools is presented as a computation ready to complete, e.g. $3+5=$, and because pressing the equals button on a calculator performs a calculation on whatever has been entered, children understand equals as meaning *compute now* rather than *is equivalent to* (Booker, 1987; Booth, 1988). Linchevski (1995) states that in the transition from arithmetic to algebra children need to move from a unidirectional view of the equals sign to a multidirectional one.

Closely related to this is the use of an equation as a process, rather than an object that can be operated on (Sfard, 1991). Children initially see equations as the description of an arithmetic process, e.g. $2\times 3+4=x$, and when presented with an equation to solve, e.g. $2x+4=10$, they also see it as the description of an arithmetic process with *guess and check* as a natural way of finding x. Even the more sophisticated strategy of solving the equation by working backwards may result from a view of equations as processes, yet this is often not revealed until students encounter equations of the kind $2x+4=3x-6$. It is no longer possible to regard the equation as the description of a process giving a result, and it is essential to view the equation as an object to be acted upon in order to solve it (Sfard & Linchevski, 1994). Herscovics & Linchevski (1994), however, present data to show that many children revert to the strategy of guess and check to solve equations of this type. The issue of operating on unknowns is another perspective on why equations with unknowns on both sides cause so many difficulties. Booker (1987) suggests that it is the shift from manipulation of numbers in order to solve for an unknown, to the manipulation of unknowns themselves that marks the entry into algebra proper.

Children's stage of numeracy is likely to be important for their understanding of expressions and equations (Irwin, 2003). Equations of the form $x+3=7$ can be solved by advanced counters through guess and check, but can be solved much more easily by part-whole thinkers able to visualise 7 as $3+4$. (See the Number Framework for a description of advanced counting and part-whole thinking (Ministry of Education, 2003b).) It can be argued that to solve equations such as $3x=15$ requires multiplicative thinking in order for a child to do more than simply follow prescribed algorithms. Furthermore equations of the kind $2x+3=11$ might require an understanding of numbers beyond simple additive part-whole or multiplicative part-whole thinking.

The Current Study

From the Numeracy Development Projects we have available diagnostic tools for assessing students' stage of numeracy (Ministry of Education, 2003a). During 2006 a group of researchers and teachers working on a Teaching and Learning Research Initiative (TLRI) project developed a diagnostic tool for assessing students solving of linear equations (Linsell, Savell, Johnston, Bell, McAuslan, & Bell, 2006). We are now using these tools to investigate the links between numeracy and students' strategies for solving linear equations and during 2007 have interviewed approximately 500 Year 7 to Year 10 students. In 2008 we are interviewing another 300 students before commencing formal data analysis. However at this point we believe it is worthwhile to share the approach we are using and some preliminary data.

In the 2006 TLRI study, the initial classification of strategies for solving equations was based on the work of Kieran (1992), whose review of the learning and teaching of algebra describes the strategies that students use. However, in this study, further strategies were added that students were observed using. For example, the use of an inverse operation (1c) (see Figure 1) on a one-step equation was considered to be different to Kieran's strategy of working backwards (3b) on multi-step equations.

It was difficult to distinguish the difference between known basic facts (1a) and inverse operations (1c) because the students often justified their answers by describing the inverse operation, when, in fact, what they had done was use a known fact. One-step equations with larger numbers were therefore used to elicit the use of inverse operations.

The strategy of working backwards was found to be not as homogeneous as had been assumed. Many students partly worked backwards and then used either known facts (3c) or guess and check (3d). When large or decimal numbers precluded the use of these strategies, the students could no longer use the working backwards strategy (3b).

The strategy of using a diagram (5) was also included. This resulted from the study's explorations of questions in context, where a number of students solved equations through direct use of diagrams.

The final classification of strategies is listed in Figure 1. It should be noted that this list of strategies is not intended to be hierarchical, as there is insufficient evidence to make such a claim. In fact, 3c and 3d are clearly less sophisticated strategies than 3b, and at present, the relative sophistication of 5 is not known.

Figure 1 Classification of strategies for solving equations

0. Unable to answer question

1a. Known basic facts

1b. Counting techniques

1c. Inverse operation

2. Guess and check

3a. Cover up

3b. Working backwards

> *3c. Working backwards then known facts*
>
> *3d. Working backwards then guess and check*
>
> 4. Formal operations/equation as object
>
> *5. Use a diagram*

(Based on Kieran, 1992; our amendments in italics)

The algebra diagnostic assessment for this study is run in two parts, a knowledge section and a strategy section.

The knowledge section is administered as a written test because supplementary questions are not required. The areas investigated in this section, together with example questions, are:

- understanding of conventions and notation

  (If $n = 4$, then what is the value of $3n$?)

- understanding of the equals sign

  (Given that $7x + 4 = 15$, find the value of the in the equation $7x + 7 =$)

- understanding of arithmetic structure

  (What is the value of $18 – 12 ÷ 6$?)

- understanding of inverse operations

  (Replace the box with +, –, ×, or ÷: If $n – 27 = 25$, then $n = 25$ 27)

- manipulating symbols/unknowns (acceptance of lack of closure)

  (Add 3 to $d – 1$).

The strategy section consists of a series of increasingly complex equations, which the students are asked to solve with an explanation of their thinking. There are 12 pairs of parallel questions – ones that are in context (that is, word problems), and ones that are purely symbolic. An example of a pair of parallel questions is:

- Our kapa haka group is made up of some Māori students and 11 Pākehā students. The whole group is divided into four equal-sized groups for practice sessions. Each practice group has 19 students in it. How many Māori students are there in our kapa haka group?

$$\frac{n+12}{4} = 18$$

The increasing complexity of the equations can be illustrated by a selection of questions from the symbolic section:

- $n - 3 = 12$ (one-step equation with single digits)
- $n + 46 = 113$ (one-step equation)
- $3n - 8 = 19$ (two-step equation)
- $5n - 2 = 3n + 6$ (unknowns on both sides)
- $2n - 3 = \frac{2n+17}{5}$ (complex structure)
- We can rearrange the equation $p = r - s$ to make r the subject, $r = \mathrm{p} + s$.

  Similarly if $v = u + at$, then $a = \ldots\ldots$ (purely symbolic equation).

The questions are presented on cards so that the more difficult questions can be omitted as required without suggesting to the student that they are not coping. Each question is read to the student to minimise the impact of reading difficulties, including difficulties with reading symbolic equations. Calculators and pencil and paper are available for the students to use, but it is stressed to the students that they may use whatever method they choose.

From the interviews we have conducted to date a number of points have become apparent:

- In all year levels there are large differences between students as to the sophistication of strategies they are able to use. Some students are unable to do more than use guess and check or counting techniques. At Year 7 the majority of students are able to use inverse operations for one-step equations, with some being able to work backwards for multi-step equations. At Year 10 the majority of students are able to work backwards for multi-step equations with only a very small proportion being able to operate formally on equations, treating them as objects.

- The strategies we have observed students using are consistent with Kieran's (1992) classification, but extend it. The strategy of working backwards is less homogeneous than previously reported. Many students are only just grasping this strategy and can use it only when the first step reveals a known basic fact to them for the next step. These students use the strategy of working backwards then known facts. Other students are prevented from fully using working backwards because of lack of knowledge of multiplication and division facts. These students use the strategy of working backwards then guess and check.
- There is a high correspondence between stage of numeracy and the most sophisticated strategy a student is able to use for solving equations. Only students who are at the level of multiplicative part-whole thinking or above are able to solve equations by working backwards or formal operations. Stage of numeracy appears to be a better predictor of the most sophisticated strategy a student is able to use than the amount of algebra teaching s/he has received.
- The most sophisticated strategy a student is able to use is extremely similar for questions that are in context or questions that are fully symbolic. In fact many students are able to use slightly more sophisticated strategies for questions that are in context.

Discussion

Instead of looking at how hard equations are to solve and whether students get them right, it appears to be more useful to look at what strategies students use. The approach used in this study is very similar to that used in the New Zealand Numeracy Development Projects, with strategy being separated out from knowledge required for strategy use. This approach will allow the classification of the students according to their most sophisticated strategy rather than the most difficult equation they are able to solve. Within numeracy teaching, students are grouped for instruction according to their most sophisticated strategy. It is suggested that a similar approach to grouping students is likely to be beneficial for teaching students to solve equations.

As has been clearly identified (Herscovics & Linchevski, 1994), students have great difficulty with formal operations, which treat equations as objects. Few students, even from year 10, were observed using formal operations, and no students were able to use formal operations who could not also use the strategy of working backwards. Consistent with the perspective of Filloy and Sutherland (1996), it is suggested that these strategies are not simply alternative approaches to solving equations, but represent different stages of conceptual development. Similarly, and not surprisingly, no students were able to work backwards who could not also use the strategy of inverse operations. Formal analysis of the data from this study will allow the construction of a hierarchy of sophistication of strategies.

In many current school programmes, little attention is paid to the students' numeracy stage when attempting to teach them to solve equations. The researchers found that even year 10 students, who have received at least two years' instruction in how to solve equations, are unable to use the more sophisticated strategies if they are below the numeracy stages of multiplicative part–whole thinking. Similarly, students are unable to use inverse operations for one-step equations if they are still at the numeracy stages that involve counting strategies. This strongly suggests that prerequisite numeracy should be considered when designing teaching programmes for algebra.

It would appear to be unnecessary to present students with purely symbolic equations in order to determine their most sophisticated strategy. Questions presented in context gave very similar information and were generally perceived by the students, particularly the younger ones, as being less threatening.

A number of teachers involved in the study have commented that the algebra diagnostic tool is more useful than the standard numeracy assessment in New Zealand (Ministry of Education, 2003a) for revealing the thinking of students at the upper end of the Number Framework. This is because numeracy assessment focuses on mental strategies (and does not value the use of algorithms), while the algebra tool has a focus on students' understanding of arithmetic structure and the nature of equations.

The preliminary data presented in this paper suggest that the approach adopted is revealing useful information about students' understanding of aspects of algebra. It is anticipated that formal data analysis will establish a hierarchy of strategies, make explicit the connections between numeracy and algebraic strategies, and clarify the role of contexts.

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