On Pattern Intuition and Symbol Intuition in Teaching and Learning Algebra

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Abstract *Geometric intuition is important in mathematical thinking. However, we rely on intuition even when we deal with algebraic problems in which no diagrams are present. Geometric thinking relies heavily on diagrams, and geometric discovery and reasoning arise naturally from intuitional observations. This would appear not to be the case for algebraic operations, which are relatively abstract. Nevertheless, if we investigate the details, we may also find that intuition plays a major role in algebraic reasoning and furthermore that it is an important task for teachers to introduce algebraic concepts and algorithms in intuitionistic ways. Here we put forth the concept of pattern intuition as it applies to algebra and discuss teaching approaches based on pattern intuition. We also provide examples to show the value of pattern intuition in mathematical reasoning.*

1. Introduction

Symbol intuition is a kind of pattern intuition. It is well established that the manipulation of formal symbol not only rests on the logic of mathematics, but also plays an important role in helping the learner construct and cognize abstract mathematical ideas. Formal symbol operations would even seem to play active roles in the process of mathematical discovery.

We think that the symbol intuition is founded mainly on algebraic operations involving formal mathematical symbols. In chemistry, H_2O is a formal symbol for water. But when H_2O appears in a chemistry equation, it takes on a special computable function. We wonder whether something similar takes place in algebra. Perhaps many obstacles to teaching algebra, in topics as varied as operations involving exponents, translating cyclic decimal fractions into quotients, perceiving “indeterminates”, and so on, might be lessened by making a comparable analysis of symbolism in algebra.

Geometric thinking is generally based on geometric intuition and geometric discovery and proof commonly make use of diagrams. At first this would appear not to be the case for algebraic operations, which are relatively abstract. Nevertheless if we investigate the details, we may also find that algebraic reasoning always relies on intuitions. It is an important task for teachers to introduce the algebraic concepts and algorithms under some intuitionistic way in algebraic curriculum. We try to establish the concept of pattern intuition in algebra and discuss the teaching approaches based on pattern intuition. We think that the pattern intuition provides an important foundation for algebraic imaginations and invention.([7],[8])

2. Graphic intuition and pattern intuition

Intuition is clearly important in teaching and learning mathematics. But it is easy to neglect the role of intuition given the prevailing influence of mathematical formalism. In teaching and learning mathematics at middle school, it is useful to find a correct route to solve problems with some directly intuitional observations. In teaching geometry , we may use intuitive diagrams or figures as a backdrop for thinking. In teaching algebra, sometimes there may be functional figures or practical equations in the questions which are helpful and intuitional. Although algebraic problems concerning algorithm principles, laws of number systems, questions of combinatorics or inequalities typically do not involve actual figures, intuition about diagrams may nonetheless be could be involved. In teaching courses, it is necessary to follow some logically intuitive way, something what we call pattern intuition. Let us observe the following examples.

Example 1. Prove the combinatorics equation $\binom{n}{m}=\binom{n-1}{m}+\binom{n-1}{m-1}$

Proof 1. (formalist approach)

$$\binom{n-1}{m}+\binom{n-1}{m-1}=\frac{(n-1)\cdots(n-m)}{m!}+\frac{(n-1)\cdots(n-m+1)}{(m-1)!}$$
$$=\frac{n(n-1)\cdots(n-m+1)}{m!}=\binom{n}{m}$$

Proof 2. (intuitional approach) We first fix one object α among the set of n elements. Then there are two cases to choose m elements from the given set of n elements. In the first case, we do not choose α, so choose m elements from the remaining n−1 elements. Then there are $\binom{n-1}{m}$ chosen ways. In the second case, we do choose α, so choose m−1 elements from the remaining n−1 elements and there are $\binom{n-1}{m-1}$ chosen ways. Therefore the total is $\binom{n-1}{m}+\binom{n-1}{m-1}$, which is just $\binom{n}{m}$ and the equation is taken into account.

The first proof, which commonly appears in textbooks used in China, is deductive; the second proof is found occasionally in supplementary textbooks. But we think the first proof is nothing more than checking, because this method is not helpful for finding the formula. The second proof serves not only to demonstrate the equation but also to help discover the formula.

Pattern intuition may be deeply rooted in our biological structures that have evolved to help humans understand the logical order of natural procedures or logical relationships of nature. In mathematics, there are not necessarily external geometric figures for pattern intuition. So in these cases, it may be helpful to make use of the more general pattern intuition. In some sense, understanding can be treated as a person's harmonious acceptance of a certain pattern intuition.

2. Pattern intuition and the formalist mathematics viewpoint

Early in ancient Greece, mathematicians established the axiom system of geometry in the Euclid's Elements. Fully-fledged formalist mathematics in the modern sense did not appear until the beginning of 20^{th} century, due to the work of David Hilbert and others. Modern formalist mathematics is based on a strict system of axioms and acceptable operations that lend mathematical reasoning a credibility that it would not otherwise have. Through formalism mathematics achieved its rock-solid foundation in logic.

The traditional view is that axiomatic systems dispense with the need for knowledge based on intuition. Indeed, intuition is not be admitted into the mathematical reasoning process. But great mathematicians have often realized that it is not possible to exclude man's intuition from any intricate mathematical reasoning.

The mathematical logician Imre Lakatos once investigated the proof of Euler's Theorem of Polyhedrons: V+F=E+2 in his works[2]. His elaborate proof is as follows. Imagine a simply connected polyhedron which can be expanded over a sphere. First imagine expanding one of the surfaces and press all the other surfaces onto the expanded one (figure 2). We could compute the amount degree of the plane angles by two distinct approaches.

There are F surfaces of the given polyhedron. Assume that the i th surface is a polygon with

s_i sides. Then the total degree of the plane angles is $\sum_{i=1}^{F}(s_i-2)\pi$, as in figure 1.

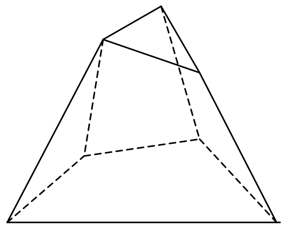


Figure 1

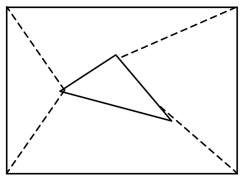


Figure 2

Assume that the expanded surface is a polygon with t sides and thus there are V-t vertices inside the expanded surface, as in figure 2. Then the total degree of the plane angles is 2(t-2)π+2(V-t) π. The total degree is equal to each other and we derive that $\sum_{i=1}^{F}(s_i-2)\pi$ =2(t-2)π +2(V-t) π. Notice that $\sum_{i=1}^{F}s_i=2E$ and so we finally have that V+F=E+2.

Lakatos noted that mathematicians accept this proof, but the geometric axioms in no way ensure one's imagination that expansion is possible. Accepting the consequences of the proof rests crucially on one's ability to imaging how one surface expands and presses other surfaces into it. By contrast, computing the degree of the plane angles is simple and plays a minor role. The proof relies on graphic transposition, which seems to be a sort of geometric operation. But the geometric operation is not a real operation. It is essentially an operation in the mind. Most people could easily accept this mental operation and believes in its reasonableness. This reasonableness derives from its association with pattern intuition.

3. Two distinct forms for mathematics

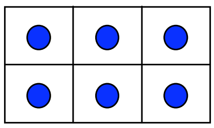
There are two distinct forms for mathematics: the academic form and the classroom form. Mathematicians present their research results within the academic form and they prove the mathematics theorems mostly by purely deduction, which imbues them with an abstract character that makes them somewhat inaccessible to the inexperienced student [6]. Classroom mathematics, however, needs to be orchestrated so as to help students understand the ideas as well as their origin.

Students often wonder whether algebraic procedures are merely arbitrary or actually reasonable means for achieving objectives. A teacher from primary school once related the story of a pupil who asked whether operations between the parentheses or those outside of the parentheses should first be carried out. In a similar way, students are sometimes puzzled about the order of arithmetical operations when no parentheses are present. Even when pupils respect the rules for order of operations, doubts may linger. Their understanding of algebraic principles is incomplete, however, if they lack intuition and familiarity with what is conventional.

Despite the fact that mathematicians established the complete axiomatic system of arithmetic, it is quite difficult to use such a system to convincingly explain a simple arithmetic law, such as the commutative law or the associative law of multiplication for rational numbers. Even though it is possible to prove these arithmetic laws by the axiomatic method, this is probably not useful for comprehending the laws of arithmetic.

Example 2. Prove the commutative law of multiplication in an intuitively friendly way.

Proof. Observe the following illustration. Counting it in row, we get 3×2, while counting it in column, then we get 2×3. We obtain $3\times2=2\times3$ and are convinced that this will hold for an two dimensional array of any size.



Example 3. Assume that *a, b, c, d* and *m* are positive real numbers. Prove the inequality $a/b<(a+m)/(b+m)$.

Proof. Suppose that we have a cup of sugar water with a mass of b grams, of which a grams is sugar. The concentration of the sugar solution is clearly a/b . If you add m grams of sugar to the mixture, then the concentration degree becomes $a+m/b+m$. We know that the water becomes sweeter. This makes it easier to accept that the inequality $a/b<(a+m)/(b+m)$ *is true.*

Many algebraic principles can be clarified by making use of pattern intuition. Teaching mathematics need not rely exclusively upon the formal proofs of academic mathematicas.

4. Four levels of mathematical cognition

George Polya once cited Dutch philosopher Spinoza concerning four levels of mathematical cognition[3]. In their view the four stages of understanding are the following:

(1) Mechanical understanding: one is able to recite a mathematical law and to accept the law without proof. He can also apply the law.

(2) Inductive understanding: One tries out the law in some simple cases and concludes that the law is correct;

(3) Logical understanding: One knows the complete proof of this law and can produce a rational explanation of the law;

(4) Intuitive understanding: One is convinced of the mathematical law and clearly understands it without question.

In their opinion the highest level of mathematical cognition is intuitive understanding.

5. The formal symbols are the advanced signs for thinking

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Human language is a kind of mediums and means developed under communications by gestures, motions and voices. The highly developed language combined with highly differentiated meanings will fit reasoning and thinking. The ideologist Bacon of the Renaissance once stated that there would be no preponderance if the human race simply relied on manipulation by hands or mentalities, whereas the preponderant instruments (language) and its auxiliaries would be developed to be the mainstream by internalized language and conceptual thinking. Sometimes the internalized language and the conceptual thinking developed parallelly, but sometimes they were confluent and influenced with each other. (see the preface of English translation of [5])

Mathematical formal symbols are highly simplified reasoning signs such as human languages and the abstract conceptions which simplifies substantialities are propitious to thinking. Mathematical formal symbols are also writing signs endowed with operating functions and obeyed certain operation laws. For example, the differential signs or integral signs obey the linear operation laws. The significances and valuations of formal symbols are realized by operation laws. In chemistry, H_2O is a formal symbol for water. But when H_2O appears in a chemistry equation, it takes on a special computable function. Modern algebra is essentially a science of studying operations of formal symbols. There exists a kind of intuition in the operations of formal symbols which is called symbol intuition. We think that symbol intuition is an important algebraic intuition.

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6. The problem of general exponent

The problem about general exponent is discussed frequently in the reform of mathematics curriculum recent time in China. It is also a difficult teaching issue for the algebraic courses in junior high school. Students are not difficult to understand the power with positive integer exponent, such as 2^2, the power with negative integer exponent, such as 2^{-2}, or the power with fractional exponent, such as $2^{\frac{1}{3}}$, and so on. But it is not so easy to explain what is $2^{\sqrt{2}}$ to high school student. In fact, the power with general exponent, such as $2^{\sqrt{2}}$, is not taught either in high school or in university. The concept of the power with general exponent is a typical the "double forgot" concept. In high school, it is impossible to precisely explain the power with general exponent, while in university it is considered not necessary to teach, because it is easy for students.

People usually considers the accurate definition of $2^{\sqrt{2}}$ should be related to the concept of limit. But this is a misunderstanding. It is difficult to understand $2^{\sqrt{2}}$ for high school students under the concept of limit, that is similar to $\sqrt{2}$, but nobody meets difficulties to understand $\sqrt{2}$. What is the difficulty for $2^{\sqrt{2}}$? We know that both $\sqrt{2}$ and $2^{\sqrt{2}}$ are irrational number. But there are more difficulties to prove $2^{\sqrt{2}}$ being irrational number than $\sqrt{2}$. To prove $2^{\sqrt{2}}$ being irrational number is almost close to solve the Hilbert 7th problem, i.e. if a is an algebraic number, $a \neq 0$ or 1, b is an algebraic irrational number, then a^b is a transcendental irrational number. In fact, four years before A.O.Gelfond and T.Schneider solved the Hilbert 7th problem in 1934, Gelfond first proved $2^{\sqrt{2}}$ is a transcendental irrational number.[4] Therefore to undersand the power with general exponent, such as $2^{\sqrt{2}}$, could not rely on accepting and understanding $2^{\sqrt{2}}$ being irrational number. It is even difficult for mathematical expert.

The operating property of $2^{\sqrt{2}}$ was ever ignored by traditional textbooks. For example, the simple operating property that $(2^{\sqrt{2}})^{\sqrt{2}} = 4$. Don't consider this is trival. We may convert this formalized cognition into some deeper truth by mathematical deduction. Observe the following proposition:

There exists two irrational numbers a and b such that a^b is a rational number.

Proof of the proposition: We can simply prove $\sqrt{2}$ is irrational. Assume that $\sqrt{2}^{\sqrt{2}}$ is irrational. Then put $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$, the conclusion is derived. Assume that $\sqrt{2}^{\sqrt{2}}$ is rational. Then put $a = b = \sqrt{2}$, the conclusion is also derived and the proof is complete.

We remark that we do not assert whether $\sqrt{2}^{\sqrt{2}}$ is irrational or not. It is very difficult to prove $\sqrt{2}^{\sqrt{2}}$ is irrational. We know that the property of the algebraic operating plays an essential role in the proof of the proposition.

7. What is symbol intuition

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The great mathematician Poincaré said in "Intuition and Logic in Mathematics" that there should be some other thing besides logic to construct arithmetic, something similar to construct geometry or other science. We may use the word "intuition" to explain this thing. We have many kinds of intuition, such as perception and imagination in the first stage. Further we generalize by induction, which is the procedure to counterdraw experiment sciences. Finally we have purely number intuition. It is the purely number intuition that enlightens and introduces the so-called analysts.

We might as well call this Poincaré's "pure number intuition" to be symbol intuition. Symbol intuition is also a kind of pattern intuition. The typical symbol intuition is the arithmetic axiom, such as, an equal amount plus another equal amount is equal. The "induction axiom" is also referred to symbol intuition. We think that mathematics symbol intuition is established on the algebraic operating.

8. The equivalence of cyclic decimal fractions to quotients

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This is another focused teaching problem in algebra curriculum. As well-known that the extension from rational number system to real number system is a transcendental extension. In general, the reasonableness of presenting a real number into an infinite decimal should depend on theory of limit. But since infinite cyclic decimals is bounded in rational number system, whether translating an infinite cyclic decimals into a quotient should also depend on theory of limit? Many researchers designed projects to fit high school students. But it is difficult to avoid theory of limit in any case. We suggest a so-called "0.999…design" as follows.

We should first accept 0.999… as a symbol of number which is provided with whole operating properties of finite decimals. Then we could simply find that 0.999…=1 by the arithmetic operating law. For example, if set α=0.999…, then 9α=10α−α=9.999…−0.999…=9 and α=1.

For a general infinite cyclic decimal, for example, β=3.789123123… =3.789 +0.123×0.001001… =3.789+0.123×0.999…/999=3789 / 1000+123 / 999000=3785334 / 999000.

We could prove the equivalence of infinite cyclic decimals with quotients by elementary method in which we merely suppose 0.999… is number symbol and satisfying algebraic operating laws.

9.Formal symbol operation plays active roles in the process of mathematical discovery

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Observe the following formal operating: $(1+2+2^2+2^3)(1+3+3^2)$=1+2+3+ 4+6+8+9+12 + 18+24+36+72. Similarly, $(1+2+2^2+\cdots)(1+3+3^2+\cdots)(1+5+5^2+\cdots)(1+7+7^2+\cdots)\cdots$ =1+2+3+4+5+6+7+8+9+10+⋯. We emphasize that this is a formal operation, because it is illegal according to the convergence theory of infinite series. But if we imagine the natural numbers as actual formal symbols in above equation, such as the formal equality $(\sum_{i=0}^{\infty} x^i)(\sum_{j=0}^{\infty} y^j)=\sum_{i,j} x^i y^j$, then it can be explained reasonably. This formal equation suggests the following strict equation:

$$(1+\frac{1}{2^s}+\frac{1}{2^{2s}}+\cdots)(1+\frac{1}{3^s}+\frac{1}{3^{2s}}+\cdots)(1+\frac{1}{5^s}+\frac{1}{5}+\cdots)=1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{5^s}+\frac{1}{6^s}+\cdots$$

This is the Euler's equation: If s>1, then $\sum_{n=1}^{\infty}\frac{1}{n^s}=\prod_p \frac{1}{1-\frac{1}{p^s}}$.

In 1859, G.B.Riemann extended the function $\varsigma(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$ to the complete complex plane and considered that this analytic function would be useful for proving the prime number theorem. Riemann investigated the zeros of this ζ-function which became a famous problem in modern mathematics.

10. An example for application of symbol intuition

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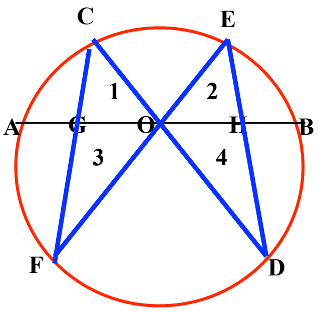
Butterfly Theorem   Let *AB* be a chord of circle with the midpoint *O*. If two other chords *CD* and *EF* go through *O* and intersect with *AB* on *G* and *H*, respectively. Then *GO*=*HO*.

Proof. Observe that there are 4 triangles. Write S_i the area of the ith triangle. Then we simply have a "foolish" equality $\frac{S_1}{S_2}.\frac{S_2}{S_3}.\frac{S_3}{S_4}.\frac{S_4}{S_1}=1$. From the theorem of sines, if there is an common angle for two triangles, then the proportion of the areas is equal to the proportion of the products of the length of the corresponding sides of the common angles. Further notice that $AG \cdot BG = CG \cdot FG$ and $AH \cdot BH = EH \cdot DH$ and the theorem will be proved.



Beautiful Butterfly

We remark that in [1] the author collected at least ten proofs for the "butterfly theorem". But no proof is as simple as this "foolish proof". The proof is benefited from the intuitional equality of formal symbols.

11. Conclusion

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Pattern intuition is more general than geometric intuition. Pattern intuition plays a particularly active role in algebra. Symbol intuition, a special kind of pattern intuition, is quite important in mathematical reasoning and proofs. Symbol intuition is established on algebraic operations. Pattern intuition and symbol intuition are also important in mathematical teaching.

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