

Algebra in Early Mathematics: a Longitudinal Intervention¹

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We report on partial results of a study aimed at introducing algebra in elementary school. We describe the main findings of the longitudinal study for a first cohort of students, from grades 3 to 5, taught by the researchers, and for a second cohort, grades 3 to 4, taught by their regular classroom teachers. The quantitative analysis of the data shows that, throughout the grades, students in the experimental group performed significantly better than the control group on written assessments developed by the research team and on standard measures.

The Early Algebra longitudinal studies we have conducted since 1997 (NSF grants #9722732, #9909591, and #0310171) have helped establish a research basis for algebra in elementary school (see Brizuela & Earnest, 2007; Brizuela & Schliemann, 2004; Carraher, Martinez, & Schliemann, 2008; Carraher & Schliemann, 2007; Carraher, Schliemann, & Brizuela, 2005; Carraher, Schliemann, Brizuela & Earnest, 2006; Carraher, Schliemann, & Schwartz, 2007; Martinez & Brizuela, 2006; Schliemann, Carraher, & Brizuela, 2007; earlyalgebra.terc.cdu).

Early Algebra (EA) refers to a general approach to early mathematics education as well as to a field of investigation. EA is about bringing out the algebraic character of elementary mathematics. In this approach, the operations of arithmetic are treated as functions of one variable (e.g. $a+3$, $7x^n$, $m\div 4$) and, later, as functions of two variables ($a+b$, mxn , $p\div q$). Likewise, compositions of functions and inverse functions are introduced early. Algebra can be introduced somewhat naturally because the basic properties of arithmetic, for example, the field axioms, are quite general, holding for any and all numbers. In order to appreciate this, students need to learn not only about specific numbers and their operations. They also need to be able to consider relations among sets of numbers, specifically, among variables. Overall, the EA approach exhibits continuity with the existing elementary mathematics curriculum. However, it aims to go more deeply into existing topics of early mathematics. One means for achieving this depth is having students express mathematical statements through various kinds of representations, particularly those involving tables, number lines and Cartesian graphs, algebraic notation, and, of course, natural language. In doing so, EA re-conceptualizes arithmetic, and,

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more generally, elementary mathematics. EA also introduces algebra itself in a somewhat novel way. EA problems often rely on background knowledge of situations and only gradually introduces standard algebraic notation and methods.

With the project “Algebra in Early Mathematics,” we are adding to the literature in the area of EA and mathematics education compelling evidence that young students (9-11 years of age) can learn algebra as an integral part of early mathematics.

Method

Participants

The study was carried out in a Boston public school in a community consisting of African-Americans and immigrants from Central America and Cape Verde. Participants in the first experimental group (cohort 1) were 26 students attending the school’s two third- to fifth-grade classrooms from 2003 to 2006. The 29 students in the second experimental group (cohort 2) attended the two 3rd to 4th grade classrooms from 2004 to 2006. The control group consisted of 30 students in the cohort immediately preceding cohort 1 (2002 to 2005) who, from 3rd to 5th grade, had been taught by the school’s regular teachers.

Procedure

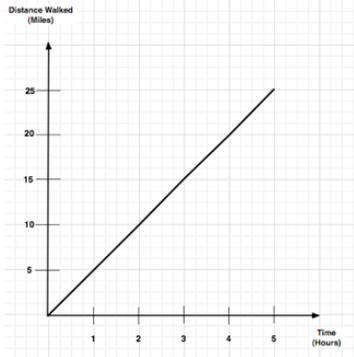
In 3rd and 4th grades, children participated each week in two 60-minute lessons and one 30-minute homework review session; in 5th grade they participated in one 90-minute weekly lesson and a 45 homework review session. Research team members implemented lessons for cohort 1 and classroom teachers conducted homework reviews. For cohort 2 the classroom teachers were in charge of all activities. A total of 50 lessons were taught in third grade, 36 in fourth, and 18 in fifth grade. All 104 lessons and corresponding homework reviews were videotaped.

At the end of each grade children in the experimental and in the control groups were given a 40-questions written assessment (50 items) that included problems designed by the research team and questions from the National Assessment of Educational Progress (NAEP) and from the Massachusetts Comprehensive Assessment System tests (MCAS). Because of the paucity of algebra-related questions on these tests, items were drawn from tests for grades 4, 6, and 8. Short written assessments were given to the students from one to four times in a semester, to evaluate their strengths and needs towards planning of the following lessons.

For the end of the year assessment in fifth grade, several problems clearly beyond the reach of control group students were added and administered to the experimental group of cohort

1. Three of these problems were later the object of an in-depth individual interview with 22 experimental group participants. The three problems are shown in Figures 1 to 3.

20. The graph below shows the distance Jose walked depending on time. Answer the following questions:
- (1) How many miles did Jose walk between 3 and 5 hours?
 - (2) How long did it take to him to walk 10 miles total?
 - (3) How long did he take to walk from 10 to 25 miles?
 - (4) How fast (in miles per hour) did Jose walk?



INTERVIEW EXTRA QUESTIONS:

- (1) Show on the graph the time between 3 and 5 hours and ask: In that time—from 3 to 5 hours—how many miles did Jose walk? Explain.
- (2) Confront the oral answer with the wrong answer he/she gave in the written test (many answered 20).
- (3) Does Jose always walk at the same pace or does he start to get faster or slower. How do you know?
- (3) Beginning at 4 hours, how much **more time** Jose will need to reach a total of 35 miles if he continues at the same pace? How do you know?

Figure 1: Graph of linear function used in written assessment and in interview with the experimental group.

15. Claudia and Adam have been playing with numbers. They each created a rule for changing any positive number you give them.

Claudia's rule: I triple the number and then add 5.

Adam's rule: I double the number and then add 12 to it.

Write their rules with algebra.

(a) Claudia's rule (use algebra): _____

(b) Adam's rule (use algebra): _____

(c) Write an equation that shows that Claudia's rule would give the same number as Adam's rule and solve the equation:

	=	
↓		↓
	=	
↓		↓
	=	
↓		↓
	=	

(d) Explain what the solution means:

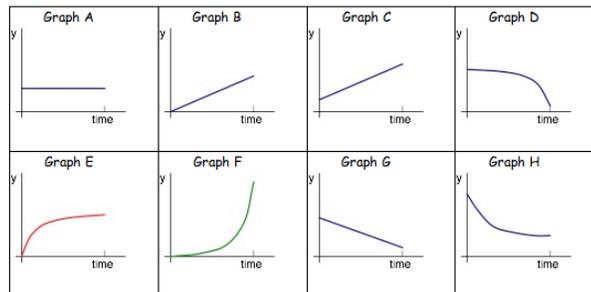
INTERVIEW EXTRA QUESTIONS:

If solution to equation is wrong, ask the child to solve it again, questioning him/her throughout the process.

- (0) Why did you choose to do the first transformation?
- (1) Explain what the solution means.
- (2) What will happen if you replace [letter student used] in the second equation with the [solution value] you have found?
- (3) Without doing the substitution, what do you think will happen if you replace [letter student used], in the first equation, with the [solution value]?
- (4) What do you think will happen if you replace [letter student used], in the first equation, with, say, [a non-solution value]?
- (5) What number will give the same result for Claudia's rule and for Adam's rule? Can you prove it to me?

Figure 2: Equation-solving problem used in written assessment and in interview.

20. Consider a water tank. The y-axis shows the height of the water in the tank. Complete the table showing which graphs match which story.



Story about the height of the water in the tank.	Graphs
The water height does not change.	
The water tank began empty.	
The water height decreases all the time.	F
	E
The water height increases all the time.	
The water height changes at a constant rate.	
The water tank is draining faster and faster.	
The water tank is draining slower and slower.	

INTERVIEW EXTRA QUESTIONS:

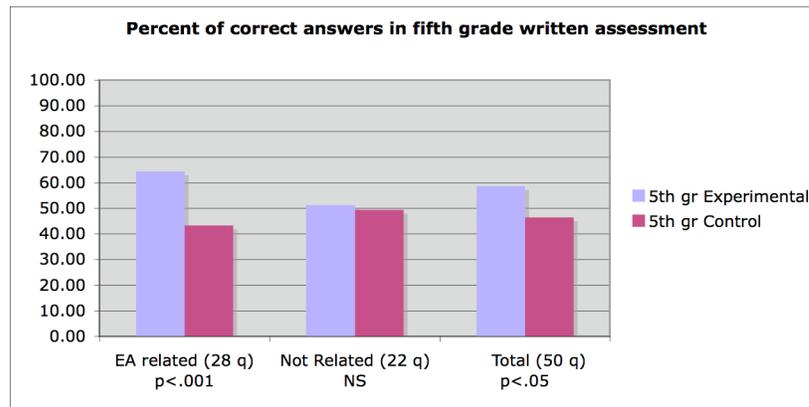
- (1) Ask the child to make a story about money that will fit graph D and to explain how the story fits the graph.
- (2) Will the same story fit graph H? Why or why not?
- (3) Will the same story fit graph G? Why or why not?

Figure 3: Problems on graphs of non-linear functions in written assessment and in interview.

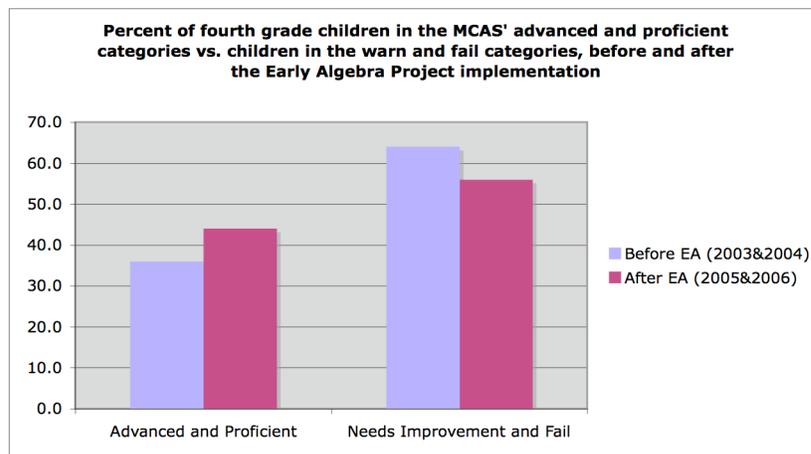
Results

General Results for Cohort 1

Of the 50 items in the written assessment given to both groups, 28 were related to the EA lessons implemented and 22 were not. Responses to each item were classified as correct or incorrect. Throughout the grades, experimental group results were significantly higher than those of the control group for items related to the EA lessons. The two groups performed equally well on items *not* related to the EA lessons. Figure 4 shows the percent of correct responses for the two groups in 5th grade.



We also evaluated students' results on the MCAS test for fourth graders before (2002 and 2003) and after (2004 and 2005) the project's intervention. Figure 5 shows that the percentage of children at Advanced or Proficient levels increased from 44% to 36%, while the percentage of those in the Needs Improvement and Fail categories decreased from 56% to 63%. The overall improvement of the experimental group was also better than the overall improvement in the school district (8.5% vs. 4.5% points). But these differences were not statistically significant.



Interpreting a Linear Function Graph

At the end of 5th grade, students in the experimental group were given the linear graph in Figure 1, showing “the distance José walked depending on time” (5 miles/hour) and asked to answer four questions. In the written assessment, only four children (18%) gave the correct answer, 10 miles, to the first question (i.e., “How many miles did José walk between 3 and 5 hours?”). They performed better in the other questions, where 10 children (45%) correctly answered “How long did it take for him to walk from 10 to 25 miles?” and 13 children (59%) answered correctly the question “How fast (in miles per hour) did Jose walk?”.

Since the expression “between 3 and 5 hours” in the first question may have misled some children to consider 4 hours as the mid-point between 3 and 5 hours and to answer 20 miles, we rephrased the question in the subsequent interview as “How many miles did Jose walk from 3 to 5 hours?” With the new wording, 59% (13 children) answered the question correctly.

In the interview a total of 19 children (86%) correctly answered “2 hours” to the question later added on “How long did it take to José to walk 10 miles total?”.

Representing an Algebra Problem and Solving Equations

For the problem in Figure 2 (see Schliemann et al., submitted), fifteen students (68.2% of all the participants) correctly represented Claudia’s rule ($3 \times N + 5$), Adam’s rule ($2 \times N + 12$), and the equation $3 \times N + 5 = 2 \times N + 12$. Three students produced wrong or incomplete rules as $\times 3 + 5$ where x appeared to mean “times”, one student wrote $3 + 5$ and $2 + 12$ and three students failed to produce a representation that could be interpreted. Ten students (45.5%) correctly solved the equation using the rules of algebra. Of the five students who correctly generated the equation but did not solve it, four proposed equal operations for each term of the equation, but failed to correctly implement them. Only one student proposed different operations for the two sides of the equation. Even though the majority of students represented the problem symbolically and nearly half could solve the equation, only one student (4.5%) explained in writing what the solution meant. It may be that the question “Explain what the solution means.”

In the individual interview, students were asked to discuss their answers to the problem and, if they had failed to do so before, to solve the equation, to explain what the solution $n=7$ meant, and to consider what would happen if 7 were to be substituted for n in the intermediate equations of the solution process. Now, the ten students who had solved the problem in the written assessment were again successful in the interview setting and eight additional students achieved the correct solution, increasing the number of correct answers to 18 (82%). Out of the 18 students who solved the equation in the interview, nine (50%) were now able to explain the

meaning of the solution $n=7$. Five other students simply re-stated the fact that n was equal to 7 and four gave explanations that were wrong or could not be interpreted. When asked what would happen if 7 were to be substituted for n in the intermediate equations of the solution process, only five (28%) of the 18 students who had solved the equation recognized that, throughout the solution process, the equality of the equation was maintained. The remaining 13 students needed to substitute other values for n and perform the computations before coming to this conclusion. For the question on what would happen if a number other than 7 were to be substituted for n in the equation, seven students (39% of those who had solved the equation) correctly stated that results on each side of the equation would be different.

While each student's interview was unique, most of their difficulties, occurring in isolation or combined, could be classified in two main groups: those related to the representation of the unknown quantity (six students) and those emerging from attempts to operate on multiple values of the unknown quantity (nine students), with three students presenting both kinds of difficulties.

Interpreting non-Linear Function Graphs

Written answer results to seven questions in Figure 3 are shown in Table 1.

Table 1. Percent of fully and partially correct responses to identification of non-linear graphs

Story about the height of the water in the tank	Fully Correct	Partially Correct
The water height does not change.	72.7	72.7
The water tank began empty.	31.8	54.5
The water height decreases all the time.	31.8	72.7
The water height increases all the time.	4.5	72.7
The water height changes at a constant rate.	4.5	36.4
The water tank is draining faster and faster.	27.3	27.3
The water tank is draining slower and slower.	27.3	27.3

The Second Cohort and Teacher Development Outcomes

One surprising finding was that the classroom teachers from grade three decided to adopt the EA lessons even after the research team had advanced, along with the experimental group, to fourth grade—surprising because the project had been designed to focus not on teacher development, but on what the students were capable of. In fact, professional development was not in the original scope of the project. The regular classroom teachers were present in the EA classes and, from time to time would work with students during problem solving sessions in the classes, that is, when the class moved from group discussions to problem solving in small groups. But they tended to play the role of observers rather than instructors. They also took

responsibility for reviewing EA homework with the students once per week. However, during cohort 1 lessons, the brunt of the teaching fell upon members of the research team.

This same phenomenon occurred at the end of grades 4 and 5. The teachers requested their own copies of the EA classes and materials so that they could carry on the work after the research team had completed its one-year accompaniment of the experimental students. Certain adjustments in the project were made to accommodate this request. Each year a member of the research team helped the teachers get their "sea legs" the first time they were to use the curriculum on their own. And one of the more experienced teachers at the school also helped the new teachers get to know about the approach adopted by the project.

Written assessments of the second cohort of students at the end of third and fourth grade, revealed that the students who were taught by the teachers were doing as well as, and sometimes better than, the students who were taught by the researchers.

In many of our teacher-researcher meetings, the current six teacher-collaborators have expressed that (a) they themselves have increased their understanding of mathematics in preparing to teach the lessons, (b) the children enjoy and understand the lessons, and (c) the EA lessons help children better deal with other topics in their regular mathematics curriculum.

Moreover, the extremely rich and comprehensive set of data we have collected in videotaped classroom lessons, written assessments, and individual interviews, in a school serving minority and first generation immigrant families in the Boston area, calls for further analysis on (a) the evolution of individual children's ways of reasoning and representing mathematical relations, as they gain access to the tools of algebra (b) how contextualized problem solving activities can help or hinder the understanding of mathematical structures and the syntax of algebra, and (c) how the teachers who taught the second cohort of students developed they own strategies and knowledge as they participated in the project and implemented our lessons.

Discussion

The evidence we have accrued is of two sorts. The quantitative analysis of the data shows that, throughout the grades, students in the experimental group performed significantly better than the control group (school peers) on written assessments developed by the research team and on standard measures. The ongoing qualitative analysis of students' work, both in and out of the classroom, tell a similar story, while providing additional insights into the particular ways students make sense of the content. In this report we describe the main findings of the longitudinal study for a cohort of students, from grades 3 to 5 and for a second cohort, from grades 3 to 4. Our results show that students in the experimental group come to (a) think of

arithmetical operations as functions rather than as mere computations on particular numbers; (b) learn about negative numbers; (c) grasp the meaning of variables, as opposed to instantiated values; (d) shift from thinking about relations among particular numbers and measures toward thinking about relations among sets of numbers and measures; (e) shift from computing numerical answers to describing and representing relations among variables; (f) build and interpret graphs of linear and non-linear functions; (g) solve algebraic problems using multiple representation systems such as tables, graphs, and written equations; (h) solve equations with variables on both sides of the equality; and (i) inter-relate different systems of representations for functions.

Specific analyses presently underway will substantially clarify children's strengths and difficulties as they are introduced to algebra. With a new NSF grant awarded to B. Brizuela and A. Schliemann, we will be evaluating the impact of the experimental group's achievement on their learning of algebra, as they progress through grades 6 to 9.

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