

From Functions to Equations in Elementary School¹

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We report on a study aimed at introducing algebra, including equation solving, from grades 3 to 5. We describe how students learn to work with functions and express relationships between quantities in narratives, tables, graphs, and equations, and how they come to understand and work with equations as the setting equal of two functions. Equations were challenging but within reach of participants.

It is now accepted that algebra has a role to play in elementary school (see Kaput, 1998; NCTM, 2000; RAND Mathematics Study Panel, 2003, Schoenfeld, 1995; and review by Carraher & Schliemann, 2007). This may appear overly ambitious given the difficulties adolescents commonly have with algebra. Algebra in middle and high school has traditionally focused on solving equations. Under such conditions (see review by Kieran, 1992), most students fail to treat letters as variables, do not operate on unknowns, do not understand that equivalent transformations on both sides of an equation do not alter its truth value, do not realize that when incorrect values are used for the unknowns in an equation the two sides of the equation are no longer equal, and cannot use algebra to solve equations with variables on both sides of the equal sign.

Functions, rather than equations, may be a more promising point of departure for introducing algebra (Schwartz & Yerushalmy, 1992), especially among young learners. The concept of function may unite seemingly disconnected topics in the early curriculum. For example, arithmetic operations are re-conceptualized as instances of functions; multiplication tables can be viewed as embodiments of the function $a \times b$ and a function such as $7x + 3$ becomes an extension of the "times 7" table (Carraher, Schliemann, & Brizuela, 2005; Carraher, Schliemann, & Schwartz, 2007; Carraher, Martinez, & Schliemann, 2008; Schliemann, Carraher, & Brizuela, 2007). With functions as a gateway into algebra, the expressions on each side of the equation can be treated as functions; the equals sign constrains each function to values in the co-domain that map to the same value in the range.

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Within a functions-based approach students need to shift their attention from particular, "actual" cases (e.g. the view that x has a determined value) to the set of possible cases. Since young students do not easily make mathematical inferences by examining algebraic expressions and deriving new ones, elementary school instruction is initially grounded in extra-mathematical situations. At first, natural language, graphs of events, and some combination of language and tables serve as the media for expressing generalizations. Algebraic notation *piggybacks* on these other representational systems and on the meaning of the modeled situations. However, sooner or later algebraic notation needs to have a "life of its own."

Here we report on a longitudinal study in 3rd to 5th grade in which the lessons focused on functions as expressed through natural language, function tables, graphs, and algebraic notation (Brizuela & Earnest, 2007; Brizuela & Schliemann, 2004; Carraher, Schliemann, & Brizuela, 2005; Carraher, Schliemann, & Schwartz, 2007; Schliemann, Carraher, & Brizuela, 2007). We began with rich problem contexts and situations involving relations between physical quantities. Students gradually learned to use letters to stand for unknown amounts and to establish correspondences across various representations of a given problem. Later, we guided them to derive conclusions directly from mathematical representations such as graphs or equations and to solve equations using by performing equivalent operations on both sides of the equation. At the end of the project, participants represented verbal problems as equations, derived new algebraic expressions from existing ones, and interpreted the results of their solutions to equations.

Method

The data are from a three-year longitudinal study of two classrooms (N=22) from an inner-city public school in Boston, MA. From the beginning of 3rd to the end of 4th grade, we implemented two one-hour weekly lessons and two half-hour reviews of homework. In 5th grade, we implemented one 90-minute weekly lesson and 45 minutes of homework review. All classroom activities were videotaped. Lessons ranged over standard topics in the curriculum such as addition, subtraction, multiplication, division, fractions, ratio, and proportion. However the topics were "algebrafied" in ways that made connections to the topics of variables, linear and non-linear functions, algebraic notation, function tables, graphs, and equations. At the end of 5th grade, students were given a 40-item written assessment and, a week later, they were interviewed about their answers. An item from this assessment that

involved an equation with a variable on each side of the equals sign is the focus of this paper. Some representative lessons are described here.

In third grade, lessons aimed at helping students to note and articulate relations among variables. For instance, in The Candy Boxes Problem (see Carraher, Schliemann, & Schwartz, 2007), the instructor showed two boxes of candies to the students and explained that one box is John's and the other is Mary's; the boxes have exactly the same number of candies inside; John has one and Mary has three additional candies resting atop their boxes. Students were asked to express John and Mary's amounts through drawing and writing. Starting with their own representations, the 3rd graders were guided to adopt literals to represent the amount of candies in a box and to write expressions for the total number of candies: $N + 1$ for John's and $N + 3$ for Mary's. In subsequent lessons they explored relationships between variables through number lines, function tables, and algebraic notation. Guess-My-Rule games, verbal problems, and the algebraic notation for functions' underlying patterns were contexts for discussions and representation of relations between variables.

As they entered 4th grade, students were already familiar with number lines for representing values and additive operations along a single dimension. We then introduced the Cartesian space by asking students to consider the statement "For each hour of work you get \$2" (The Human Graph; see Schliemann & Carraher, 2002). At first they kept track of values for each of two variables on two parallel number lines drawn on the floor. The number lines then became the axes of a coordinate space via a rotation of one of them by 90 degrees. Students "plotted themselves" at points in the plane consistent with the statement. Later they worked with graphs on paper or on the blackboard, interpreting graphs and building function tables and graphs to express the relations of earnings per hour, candy bars per person, and distance per time, first presented to them as verbal statements.

At the end of 4th grade we introduced problems involving the comparison of two linear functions. In The Wallet Problem (Carraher, Schliemann, & Schwartz, 2007), students were told: "Mike and Robin each have some money. Mike has \$8 in his hand and the rest of his money is in his wallet. Robin has altogether exactly three times as much money as Mike has in his wallet. How much money could there be in Mike's wallet? Who has more money?" The students produced a table of values to capture the possibilities in the problem according to the amount in the wallet, plot the functions, and made sense of the intersection of the graphs as well as the regions where one graph surpassed the other. They expressed Mike's

amount $n + 8$ and Robin's as $3n$ and realized that the amounts are the same only if there are four dollars in Mike's wallet. This introduces them to the idea that equations are special cases where each function produces the same value.

Fifth grade began with ten lessons about equations, shown in Table 1.

Table 1: Summary of lessons on producing and solving equations.

Word Problems and Activities	Pedagogical Aims
Lesson 1: Anna went to the arcade with some money. She first spent five dollars playing video games. Later that day she won a prize where they doubled her money. The same day, Bobby went to the arcade with ten dollars. When he got there, his mother gave him thirty more dollars. Afterwards, he spent half of all of his money playing video games. Represent Anna's and Bobby's money at the arcade at the end of the day.	To write algebraic expressions to represent known and unknown amounts, in this case $2(n - 5)$ and 20 , to represent Anna's and Bobby's amounts.
Lesson 2: Anna went to the arcade with some money. She spent \$8 on video games. Later she won a prize that doubled her money. The same day, Bobby went to the arcade with \$16. There, his mother gave him \$20 more. Afterwards, he spent half of all of his money playing video games. At the end of the day, Anna and Bobby counted their money and discovered that they had the same amount of money.	To write and solve an equation with a variable on one side of the equals sign and relate the solution to the story context. The equation was of the form $2(n - 8) = 18$. Students are introduced to rules for solving equations through syntactic manipulations using the template in Figure 1.
Lesson 3: Students enact a situation involving specified numbers of loose candies as well as unknown amounts of candies in tubes and boxes.	To write and solve equations of the form $3x + y + 6 = x + y + 20$, where x stands for amount of candies per tube and y for amount of candies per box.
Lessons 4 and 5: Assessment I and Review of students' answers to assessment.	To assess children's progress and help them identify, discuss, and overcome difficulties.
Lesson 6: "Unsolving equations": Students produce equations from a solution.	To emphasize how valid transformations can be made to both sides of an equation without arriving at the canonical form, $x = k$.
Lesson 7: Starting with a solution (e.g., $q = 10$), equivalent equations are created by transforming both sides of the equation by a set rule generated by the students.	To emphasize that performing equivalent transformations on both sides of an equation does not change its solution.
Lesson 8: Students solve a series of equations with variables on both sides of the equals sign.	To develop proficiency in solving equations, manipulating familiar symbols without relying on a story problem and template.
Lesson 9: Elizabeth and Darin each have some money. Elizabeth has \$40 in her wallet and the rest of her money is in her piggy bank. Darin has, altogether, exactly five times as much money as Elizabeth has in her piggy bank. Elizabeth's total amount of money is equal to Darin's total amount of money. Write an equation showing that Elizabeth's total amount of money is equal to Darin's total amount of money. Solve the equation. How much money does Elizabeth have in her piggy bank?	To set up and solve an equation with a variable on both sides of the equal sign ($n+40=5n$) and relate the solution to the story context.
Lesson 10: Assessment II.	To assess children's progress.

In lesson 2 students represented a verbal problem as an equation and were guided to solve it using the template in Figure 1, proceeding in a downward fashion while registering the operations performed on each side of the equation. The recorded operations served as the basis of discussions about the steps in the process of producing a solution of canonical form, $n = k$. The students then solved a similar problem in small groups.

Anna went to the arcade with some money. She spent \$8 in video games. Later she won a prize where they doubled her money.

The same day, Bobby went to the arcade with \$16. There, his mother gave him \$20 more. Afterwards, he spent half of all of his money playing video games.

At the end of the day, Anna and Bobby counted their money and discovered that they had the same amount of money.

Here is the equation showing that Anna and Bobby had the exact same amounts at the end of the day:

$$2(n - 8) = 18$$

How much money did Anna have when she arrived at the arcade?
 Or: How can you figure out what n is equal to?

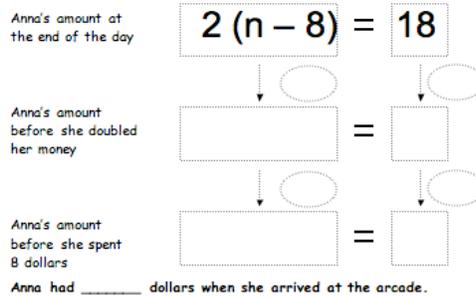


Figure 1: The template introduced in Lesson 2

In lesson 3 (see Brizuela & Schliemann, 2004) containers (enclosed in opaque tubes and boxes) were displayed on two tables along with loose candies in transparent bags. The containers thus embodied variables: each tube was to have the same amount of candies as each other tube; likewise for boxes. The loose candies embodied constants. On the left were 3 tubes, 1 box, and 6 loose candies, said to belong to one child. On the right side were one tube, 1 box, and 20 loose candies. After the students discussed the possible amounts they were informed that the two children in the problem had the same amount of candies. The problem thus embodied the equation $3x + y + 6 = x + y + 20$, where x is the amount of candies per tube and y is the amount of candies per box. Students were asked to write and solve an equation that expressed the problem. Discussions focused on the equivalence of $3x$, "3 times x ", and $x + x + x$, the irrelevance of y 's value, and the idea that the equality is maintained after performing equal changes to both sides of the equation. Students also solved other written equations with a similar structure but no word problem.

In lesson 6 we introduced an activity for "unsolving equations" (Schwartz, 2005). Students were given the solution $w = 21$, instructed to implement the transformations listed in each row of the template and asked to decide whether or not, at each step, the unknown amount remained the same. In a second similar activity (lessons 6 and 7), a student in a group suggested a solution (e.g. $n = 4$) and passed it to a peer who proposed an operation to perform

on to each side of the equation and then carried it out. When all group participants had taken a turn, the solution (given at the start) was applied to the final equation.

Students solved a series of equations in Lesson 8, without word problems or a template. Lesson 9 was similar to lesson 2, but involved variables on both sides of the equation. Lesson 10 was dedicated to a written assessment and review of students' answers.

In the last three lessons of the project, students compared pairs of linear functions introduced through extra-mathematical contexts: phone plans, payment plans for shoveling snow, and two trains traveling on a track. In a phone plans problem, one plan had a monthly fee plus a charge based on the number of minutes spoken. The second charged only by the minute, but the rate was higher than for the first plan. Students filled in tables for varying amounts of calling and then produced, for each plan, graphs of the monthly charges by minutes spoken. In discussing the graphs and tables, they arrived at the conclusion that the plan with a monthly fee was more economical for those who spoke beyond a certain minimum per month and also noted that the charges would be the same for the special case where the graphs intersected.

Two weeks after this last lesson, students were given a written assessment in which they were asked to compare functions. Four months later, at the end of the school year, in a final 40-item written assessment, students were asked to solve the problem in Figure 2 and, a week later they were individually interviewed, discussing their written answers.

15. Claudia and Adam have been playing with numbers. They each created a rule for changing any positive number you give them.

Claudia's rule: I triple the number and then add 5.

Adam's rule: I double the number and then add 12 to it.

Write their rules with algebra.

(a) Claudia's rule (use algebra): _____

(b) Adam's rule (use algebra): _____

(c) Write an equation that shows that Claudia's rule would give the same number as Adam's rule and solve the equation:

	=	
▼		▼
	=	
▼		▼
	=	
▼		▼
	=	

(d) Explain what the solution means:

Figure 2: Problem used in the written assessment

Results

Fifteen students (68.2%) correctly *represented* the two rules and the equation $3 \times n + 5 = 2 \times n + 12$; ten students (45.5%) correctly *solved* the equation using algebra. Of the five students who generated the equation but did not solve it, four proposed equal changes for each side of the equation but made errors of implementation. Their mistakes included proposing to divide by $2n$ rather than to subtract $2n$ from each side and writing 2 as a result of $2n - n$. Only one student proposed different operations for the two sides of the equation. Somewhat surprisingly, only one student (4.5%) explained what the solution meant.

In the individual interview, students discussed their answers to the problem and, if they had failed to do so before, tried to solve the equation, explain what the solution $n = 7$ meant, and consider what would happen if 7 were to be substituted for n in the intermediate equations of the solution process. Under these circumstances eight additional students achieved the correct solution, raising the number of correct responses to 18 (82%). One of these students solved the equation with no help from the interviewer, four needed a small amount of help, and three needed substantial support. Nine of the 18 students (50%) explained the meaning of the solution $n = 7$ during the interview. But only five of the 18 students (28%) recognized without computing that the equality would be maintained if 7 were to be substituted for n in the intermediate equations. The other students had to substitute other values for n and perform the computations before coming to this conclusion. For the question asking what if a number other than 7 were to be substituted for n , seven students (39%) correctly stated that results on each side of the equation would be different.

Students' difficulties were of two sorts. Six students had difficulty representing the unknown quantity. Nine students made errors while operating on multiple values of the unknown quantity; they attempted to subtract either the parameter or the unknown from an expression (e.g. $3a-3 \rightarrow a$). Three students displayed both kinds of difficulties.

Discussion

Students' performance at the end of the three-year intervention was generally encouraging. The present data raise doubts about previous conclusions that equations with variables on both sides of the equality are beyond the reach of young students. Despite the well documented difficulties older students have using letters in algebra, the majority of our students appropriately used letters to represent variable amounts. Moreover, while previous research has shown that even high school students may not realize that the left and right sides

of the equation are no longer equivalent when an incorrect value is used in an equation, seven of our fifth graders explicitly stated that using a value other than the solution to the equation would destroy the equality.

Syntactical errors with coefficients by some of our 5th graders were similar to those previously found among middle and high school students. Most students had no trouble subtracting 5 from each side of the equation $3n + 5 = 2n + 12$, yet some had difficulties working with $2n$ and $3n$.

Indeed, the transition from the semantics of the problem to the syntactic rules of algebra is a rather challenging endeavor. Despite these difficulties, our data show that young students can use algebraic notation to represent problems and succeed in solving equations with variables on both sides of the equality.

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