THE RELATION BETWEEN AFFECT AND THE PROVING PROCESS

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In this largely theoretical paper, we discuss the relation between a kind of affect and aspects of the proving process. We begin with affect as described in the mathematics education literature, but soon narrow our focus to a particular kind of affect – nonemotional cognitive feelings. We then mention the position of feelings in consciousness because that bears on the kind of data about feelings that students can be expected to be able to express. Next we introduce the idea of behavioural schemas as enduring mental structures that link situations to actions, in short, habits of mind, that appear to drive many mental actions in the proving process. This leads to a discussion of the way feelings can both help cause mental actions and also arise from them. Then we briefly describe a design experiment – a course intended to help advanced undergraduate and beginning graduate students improve their proving abilities. Finally, drawing on data from the course, along with several interviews, we illustrate how these perspectives on affect and on behavioral schemas appear to explain, and are consistent with, our students' actions.

1. INTRODUCTION

The proving process plays a significant role in both learning and teaching many advanced mathematical topics, such as abstract algebra and real analysis. In studying such topics, advanced undergraduate and graduate mathematics students not only read proofs, but also supplement those proofs with their own subproofs. Their teachers not only explain proofs, but also ask them to construct proofs, especially as a way of assessing the students' understandings. As a consequence, we are designing a course to help advanced undergraduate and beginning graduate students improve their proving abilities We are also exploring the proving process, attending to whatever contributes to that process, in particular, affect. In this paper, we focus on a particular kind of affect -- nonemotional cognitive feelings.

2. PERSPECTIVES ON AFFECT

2.1 Beliefs, attitudes, emotions, and values

Affect is often seen as separate from, but related to, cognition. We start with McLeod's (1992) treatment of affect as having three main aspects: (1) *beliefs*, for example, the belief that mathematics is based on rules or that teaching is telling; (2) *attitudes*, for example, dislike of geometric proving or enjoyment of problem solving; and (3) *emotions*, for example, the joy of, or frustration with, solving nonroutine problems. Beliefs, attitudes, and emotions are described by McLeod (1992) as listed in increasing intensity and decreasing stability, with emotions being the most intense and changing the most rapidly.

There are many interesting connections between affect, in the above view, and teaching, learning, and doing mathematics. While teaching a college algebra course, the first author occasionally gave group quizzes on which students were to solve moderately nonroutine problems. As she walked around answering questions and facilitating the group work, one student was not contributing to her group's solutions, but rather was very quietly repeating to herself, "I hate this stuff. I hate this stuff." We infer that this student was expressing a negative emotion, and hence, dividing her attention between the emotion and the problem at hand. This very probably overburdened her working memory and interfered with trying to solve the problem and contributing to the group's efforts.

In the above example, we suggest it was very likely that the details of the algebra problem and how it was being attempted were, more or less, immaterial to the student. Thus, affect, in this case an emotion, was associated in a rather "large-grained" way with doing mathematics, that is, with attempting to solve almost all algebra problems. A good deal of the research on the relationship between affect and mathematics seems to have taken a large-grained approach (DeBellis & Goldin, 1997, 1999, 2006; Hannula, Evans, Philippou, & Zan, 2004; McLeod, 1992). However, in this paper we take a finer-grained approach. We analyze the relationship between specific kinds of affect and small aspects of the proving process.

Here is an example that seems to call for a finer-grained view of the proving process, and also for a somewhat expanded perspective on affect. During tutoring by the third author, a student, Sofia, from our recent course on proving, had produced what we call the *formal-rhetorical* part of a proof (Selden & Selden, in press), that is, the part that depends only on unpacking and using the logical structure of the statement of a theorem and associated definitions. For someone experienced in proving, this part does not depend on a deep understanding of, or intuition about, the concepts involved or on genuine problem solving in the sense of Schoenfeld (1985, p. 74). At this point Sofia needed an idea to continue proving the theorem. We infer that Sofia was aware of needing an idea because she had had experience writing the formal-rhetorical part of proofs and would have understood that her proof was not complete. As she had done on several previous occasions, Sofia then suggested an idea that seemed to us not to be rationally connected to the problem at hand. We have come to call such actions "unreflective guesses." The tutor then needed to take an action – even doing nothing would likely have been interpreted by Sofia as an action or as having meaning. But to act usefully the tutor needed to understand why Sofia made her peculiar suggestion. We believe this suggestion was triggered by some kind of affect, but a kind not among the three major aspects previously discussed. On viewing the video, we found no evidence of an emotional response, and we have no reason to believe Sofia had an underlying belief or attitude related to this incident.

We believe Sofia's suggestion was triggered by a feeling of confusion or by a feeling of not knowing what to do next.

The above example of Sofia suggests that it would be useful to expand one's view of affect somewhat, and that particular aspects of affect may be linked to specific parts of the proving process. DeBellis and Goldin (1997, 1999, 2006) have added a forth main aspect to affect, namely, *values*. While this addition is a reasonable extension of affect, it does not account for Sofia's behavior. DeBellis and Goldin also point out that "affect in not auxiliary to cognition; it is centrally intertwined with it" (Hannula, et al., 2004). They see "affect as a highly structured system that encodes information, interacting fundamentally – and reciprocally – with cognition" (1999, p. 2-249).

2.2 Feelings

DeBellis and Goldin, as well as Ortony, Clore, and Collins (1988) see feelings as having an appraisal value that can either be positive or negative. In addition, Clore has considered "feelings-as-information." DeBellis and Goldin (2006) refer to "emotional feelings," however Damasio (2003) distinguishes between emotions and feelings – the former being public and the latter being private (p. 27). "Emotions play out in the theatre of the body. Feelings play out in the theatre of the mind" (p. 28). Most of the emotions that Damasio considers, such as joy and sorrow, as well as some less intense emotions, are a complex collection of chemical and neural responses to a stimulus that may produce bodily changes, such as changes in one's heart rate, temperature, and so forth (p. 53).

Because this distinction between feelings and emotions may be somewhat counterintuitive, we paraphrase one of Damasio's salient examples. As doctors were placing tiny electrodes in the mesencephalon of the brain stem of a 65-year old woman suffering from Parkinson's disease, the patient abruptly stopped her ongoing conversation, began to look sad, and a *few seconds later* suddenly began to cry. She said she had no energy to go on living. The doctors quickly removed the offending electrode and the sobbing stopped as abruptly as it had begun, and the sadness vanished from the woman's face. "The sequence of events in this patient reveals that the emotion sadness came first. The feeling of sadness followed … Once the stimulation ceased these manifestations waned and then vanished. The emotion disappeared and so did the feeling. … The importance of this rare neurological incident is apparent. … As thoughts normally causative of emotions appear in the mind, they cause emotions, which give rise to feelings, which conjure up other thoughts that … amplify the emotional state. … More emotion gives rise to more feeling, and the cycle continues until distraction or reason put an end to it. … By the time all these sets of phenomena are in full swing … it is difficult to tell by introspection which came first. This woman's case helps us see through the conflation [between emotions and feelings]." (Damasio, 2003, p. 57). Thus, emotions and feelings are distinct – one physical and the other mental. Each can sometimes elicit the other, and a particular feeling may, or may not, become lastingly associated with a particular emotion.

In discussing feelings, Clore (1992) distinguishes emotional feelings from nonemotional cognitive feelings, such as a feeling of knowing. For example, one might experience a feeling of knowing that one has seen a theorem useful in a current problem, but not be able to bring it to mind immediately. Such feelings of knowing can guide cognitive actions because they can influence whether one continues a search or aborts it (Clore, p. 151).

Some nonemotional cognitive feelings, different from a feeling of knowing, are a feeling of familiarity and a feeling of rightness. Mangan (2001) distinguishes the two. Of the former, he says that the "intensity with which we feel familiarity indicates how often a content now in consciousness has been encountered before," and this feeling is different from a feeling of rightness (Mangan, 2001, Section 1, Paragraph 3). It is rightness, not familiarity that is "the feeling-of-knowing in implicit cognition" (Mangan, 2001, abstract). Rightness is "the core feeling of positive evaluation, of coherence, of meaningfulness, of knowledge" (Mangan 2002, Section 1, Paragraph 11).

Some feelings are not only nonemotional, but they also have no sensory component. They are non-sensory experiences, and are, for example, not red or hot. Such feelings do not have a verbal component, but are sometimes described in words. "The feeling of familiarity is not a color, not an aroma, not a taste, not a sound. It is possible for the feeling of familiarity to merge with, or be absent from, virtually any sensory content found on any sensory dimension" (Mangan, 2001, Section 1, Paragraph 7). From the point of view of problem solving or proving, another important non-sensory experience is a feeling of rightness (James, 1890; Mangan, 2001).

In regard to a feeling of rightness, Mangan (2001, Section 6, Paragraph 7) says "people are often unable to identify the precise phenomenological basis for their judgments, even though they can make these judgments with consistency and, often, with conviction. To explain this capacity, people talk about 'gut feelings,' 'just knowing,' hunches, intuitions."

We focus on feelings that are often not intense, such as feelings of knowing, of caution, of familiarity, of confusion, of not knowing what to do next, of rightness/appropriateness, of rightness/direction, or of rightness/summation. Such feelings are non-sensory experiences that, at any particular moment, can pervade one's whole conscious field. They may be rather "vague" and not easily noticed or focused upon, but can influence one's actions (Mangan, 2001, Section 1, Paragraph 4), including actions that are part of the proving process.

Feelings of rightness can give direction, be summative, or suggest appropriateness. Of a feeling of rightness/direction, Mangan says, "In trying to solve, say, a demanding math problem, [a feeling of] rightness/wrongness gives us a sense of more or less promising directions long before we have the actual solution in hand" (2001, Section 6, Paragraph 3). A feeling of rightness/summation can integrate and evaluate "large sets of information necessary for the problem-solving [or theorem proving] processes" (Damasio, 2001, p. 177).

It appears that often at the end of reading or writing a proof, short-term memory is inadequate to hold sufficient detailed information to allow a rational judgment of whether the proof is correct. However, something must cause an individual to decide his own, or someone else's, proof is correct. We see a (summative) feeling of rightness as playing a major role in such decisions.

2.3 Our perspective on affect

We see affect as having five main aspects: beliefs, values, attitudes, feelings, and emotions. Because one *has* or *experiences* these aspects of affect, we regard them as part of what one might call the passive mind. In contrast, cognition, and in particular the proving process, is part of what one might call the active mind, referring to mental *acts*, for example, producing inner speech or vision, or bringing something to mind. Beliefs, values, and attitudes, like knowledge, are lasting in varying degrees, and can be "activated," but they are not themselves continually experienced, that is, they are not usually part of consciousness. In contrast, feelings and emotions are experiences, that is, they are part of consciousness. We follow Damasio (2003) in taking emotions to be physically embodied and feelings to be mental states. Within feelings, some are associated with the senses, for example, a feeling of pain, and some are non-sensory, for example, the feeling that a certain argument is correct. Also, some feelings are associated with emotions, for example, a feeling of joy or sorrow, and some are not, for example, a feeling that an argument is correct.

Finally, although all of affect can provide information for cognition, certain non-sensory feelings are cognitive in that they are about, or relate to, parts of the cognitive process. For example, the feeling in constructing a proof that one is "on the right track" is a cognitive feeling. In this paper, we focus on nonemotional cognitive feelings and how they might interact with the proving process.

3. OBTAINING INFORMATION ABOUT FEELINGS

Although a person's feelings are not directly observable, they are conscious and potentially reportable. However, not everything experienced is likely to be reported, for example, in "think aloud" episodes. This is because consciousness has different aspects, some of which are not easily focused upon in order to formulate a report.

In discussing consciousness, we are referring to phenomenological consciousness, that is, to the subjective experiences that everyone is aware of. For example, this includes experiences arising from the senses (vision, hearing, etc.), one's own speech, and other physical actions, as well as the corresponding inner versions (inner vision, inner speech, etc.). In addition, consciousness includes more subtle experiences, often without a sensory component, such as the feeling that a proof is correct.

At any one moment, a person's conscious field can accommodate a number of experiences, but this capacity is quite limited. Perhaps because of this limited capacity, some sensory experiences are focused upon and are of high resolution, and many other sensory experiences are more peripheral, vague, and of a lower resolution. If new sensory experiences are focused upon, then the earlier focal sensory experiences immediately become more peripheral and vague. In contrast, non-sensory experiences pervade the entire conscious field, but are also often vague and of low resolution (James 1890, Mangan 2001).

In think-aloud problem solving, the reporting process has a more limited capacity than consciousness, and there is no time for reflection, so reports are likely to be restricted mainly to what is being focused upon – probably the experiences of the problem solver's own actions in the problem-solving process. Except in rare cases of very intense feelings, we expect they would only be mentioned in response to a direct question. Hurlburt has developed a "beeper" technique for obtaining "snapshot" reports of the contents of a person's consciousness (Hurlburt & Schwitzgebel, 2007), but here again we suspect direct questions would be required to get reports on feelings.

4. THE PROVING PROCESS

The psychological process of constructing a proof of a theorem is much more complex than the resulting proof, and no doubt partly occurs outside of consciousness. Here we suggest a theoretical perspective for only a small part of the proving process. A person with at least a little proving experience is likely to begin to recognize, although perhaps not name or focus on, particular kinds of situations in an uncompleted proving process, that he/she often links to particular kinds of mental actions. Eventually such a <situation, action> pair may have a lasting, automated quality, and the warrants originally needed to justify the action may no longer play a conscious role.

For example, Mary, an advanced graduate student in mathematics described to us such a <situation, action> pair in proving that a compact set A in R^n is bounded. Mary and two graduate student collaborators assumed A to be unbounded and were able to construct an open cover of A that had no finite subcover. They then immediately observed that this contradicted the compactness of A and that this proved the result. Mary, who had never studied formal logic, reported that, upon finding the cover had no finite subcover, she immediately knew the result was proved. She did not reflect on the logical structure of what had transpired in an effort to explicitly warrant that the proof was complete. The other two students also did not appear to require reflection or an explicit warrant.

We infer that each student had recognized the situation as similar to one that they had experienced several times previously involving a hypothesis, a conclusion assumed false, and a resulting contradiction. The action was simply deciding the result had been proved. The link between the situation and the action appeared to be automatic and not to require reflection or a warrant. We see this as due to the students' extensive proof constructing experience. Many less experienced students require considerable reflection and wonder needlessly what should be contradicted. Such <situation, action> pairs are common and play a significant role in the proving process.

We regard <situation, action> pairs that occur regularly as the enactment of enduring mental structures that we call *behavioral schemas*. These are partly procedural knowledge, that is, knowing *how* to act, and partly what Mason and Spence (1999) have called "knowing *to* act in the moment." We view behavioral schemas as belonging to a person's knowledge base. The action produced by the enactment of a behavioral schema might be simple, as in the above example, or complex, such as a procedure consisting of several smaller actions, each produced by the enactment of its own behavioral schema. When viewed in a large grain-size, behavioral schemas might also be regarded as habits of mind (Margolis, 1993).

It appears that consciousness plays an essential role in understanding the enactment of behavioral schemas. This is reminiscent of the role it plays in reflection. It is hard to see how reflection, treated as re-presenting experiences (Steffe & Thompson, 2000), could be possible without first having had the experiences, that is, without first being conscious of them. We offer the following six-point theoretical sketch of the enactment and origin of behavioral schemas.

1. Within very broad contextual considerations, behavioral schemas are immediately available. They do not normally have to be remembered, that is, searched for and brought to mind.
2. Behavioral schemas operate outside of consciousness. One is not aware of doing anything immediately prior to the resulting action.
3. Behavioral schemas tend to produce immediate action, which may lead to later action. One becomes conscious of the action resulting from a behavioral schema as it occurs or immediately after it occurs.
4. A behavioral schema that would produce a particular action cannot pass that information, outside of consciousness, to be acted on by another behavioral schema. The first action must actually take place and become conscious in order to become information acted on by the second behavioral schema. That is, one cannot "chain together" behavioral

schemas in a way that functions entirely outside of consciousness and produces consciousness of only the action of the last behavioral schema. For example, if the solution to a linear equation would normally require several steps, one cannot give the final answer without being conscious of some of the intermediate steps.

5. An action due to a behavioral schema depends on conscious input, at least in large part. In general, a stimulus need not become conscious to influence a person's actions, but such influence is normally not precise enough for doing mathematics.
6. Behavioral schemas are acquired (learned) through habituation. That is, to acquire such a schema a person should carry out the appropriate action correctly a number of times. Changing an incorrect behavioral schema requires similar, perhaps longer, practice.

Elaborations of, and justifications for, this theoretical sketch can be found in Selden and Selden (2008).

5. THE COURSE

The setting was a design experiment, consisting of a Modified Moore Method course (Jones, 1977; Mahavier, 1999), whose sole purpose was to improve the proving skills of beginning graduate and advanced undergraduate mathematics students. The course was consistent with a constructivist point of view, in that we attempted to maximize students' proof writing experiences. It was also somewhat Vygotskian in that we represented to the students how the mathematics community writes proofs.

The students were given self-contained notes consisting of statements of theorems, definitions, and requests for examples, but no proofs. The students presented their proofs in class, and the proofs were critiqued. Suggestions for improvements in their notation and style of writing were also given. There were no formal lectures, and all comments and conversations were based on the students' work. The course carried three credits and lasted one semester. It met for one hour and fifteen minutes twice a week, making 30 class meetings altogether. We have now taught two, of a projected eight iterations of the course. There are two versions of the course, and either or both can be taken for credit. One version covers some basic ideas about sets, functions, real analysis, and semigroups. The other version covers sets, functions, some real analysis, and topology. The specific topics covered are of less importance than giving students opportunities to experience as many different types of proofs as possible.

6. FEELINGS AND THE PROVING PROCESS

6.1 Nonemotional cognitive feelings and the enactment of behavioral schemas

In constructing a proof, when a person recognizes a situation and enacts a behavioral schema, yielding a mental action, a feeling of rightness or appropriateness may soon be experienced. For example, such a feeling can be generated internally by finding a warrant for the action while reflecting on the proof in an effort to establish its correctness. Such a feeling can also be generated by an external authority, such as a teacher certifying that the proof is correct. If the feeling occurs several times when the same behavioral schema is enacted, then the appropriateness of enacting the behavioral schema may be enduringly associated with the situation. We suggest that such an association of a feeling with a situation will increase the probability of enacting the schema in future encounters with the situation.

6.2 Mary develops a feeling of rightness/appropriateness

The following report is based on separate video-recorded interviews with Mary, the above mentioned advanced mathematics graduate student (Section 4), and her real analysis teacher, Dr. K. The interviews concerned events that took place two years earlier, when Mary was taking both a pilot version of our proofs course (Section 5) and Dr. K's real analysis course. In the homework for Dr.K's course, Mary needed to prove many statements that included phrases like "For all real numbers *x*," where *x* represented a variable -- the situation. In her proofs, Mary needed to write something like "Let *x* be a number," where *x* represented an arbitrary, but fixed number -- the action. Dr. K often discussed Mary's proofs with her, and in particular,thought she carried out this action based on his authority.

When Mary was interviewed about this <situation, action> pair she said the following:

> M: At that point [early in Dr. K's real analysis course] my biggest idea was, well he said to "do it," so I'm going to do it because I want to get full credit. And so I didn't have a real sense of why it worked.
>
> I: Did you have any feeling … if it was positive or negative, or extra …
>
> M: Well, I guess I had a feeling of discomfort …
>
> I: Did this particular feature [having to fix *x*] keep coming up in proofs?
>
> M: … it comes up a lot and what happened, and I don't remember [exactly] when, is that instead of being rote and kind of uncomfortable, it started to just make sense … By the end of the semester this was very comfortable for me.

We suggest Mary developed both the behavioral schema and the associated feeling of appropriateness only after executing the <situation, action> pair numerous times. In early executions of this <situation, action> pair, Mary carried it out partly based on Dr. K's authority. However, she reported that, after completing each such proof, she convinced herself that the proof was correct. Only after repeatedly executing this <situation, action> pair, and convincing herself that the individual proofs were correct, did she develop a feeling of appropriateness. For another example, see the case of Edward in Selden, Selden, and McKee (2008).

6.3 Sofia's reaction to not having an idea

In Section 6.2, we had access to a rare first-person account that illustrates Mary's joint development of a feeling of appropriateness and a behavioral schema. In contrast, in this section, we describe how *we* used our theoretical perspective about feelings and behavioral schemas *to infer* that Sophia's progress was blocked by multiple enactments of an "unreflective guess" schema, and how our intervention appears to have weakened that schema and improved her prospects for later success in graduate school.

Sofia, the above mentioned first-year graduate student (Section 2.1), was taking the second iteration of our proofs course (Section 5). Our analysis is based on third-person information, that is, observations and video recordings of Sofia's class participation and of her proof constructions during seven individual tutoring sessions with the third author.

Sofia was a diligent student; however as the course progressed, an unfortunate pattern emerged in her proving attempts. When she did not have an idea for how to proceed, she often produced what one might call an "unreflective guess" only loosely related to the context at hand, after which she could not make further progress. We do not see the common practice of guessing a direction and then adjusting according to its fruitfulness as generally inappropriate. However, although we could sometimes speculate on the origins of Sofia's guesses, we could not see how they could reasonably have been helpful in making a proof, and she did not seem to reflect on, or evaluate, them herself.

We inferred that, in such unreflective guessing, Sofia was enacting a behavioral schema that depended on a feeling. She was recognizing a situation, that is, that she had written as much of a proof as she could, and had a feeling of not knowing what to do next. This situation was linked in an automated way to the action of just guessing any approach that usually was only loosely related to the problem at hand without much reflection on its likely usefulness. We also thought that the enactment of an "unreflective guess" schema was very likely to increase Sofia's feeling of not knowing what to do next, and thus, lead to another unreflective guess, or to abandoning her proof attempt. We judged that without our addressing this unfortunate behavioral schema, Sofia would not make progress on the problem-oriented part of proofs.

We began to suspect Sofia might have a persistant difficulty when she volunteered to present the following "proof" on the fifth day of class. Only her first and last lines could reasonably be part of a proof.

Thm 4: For any sets A, B and C, if $A \subseteq B$ then $(A \cap C) \subseteq (B \cap C)$.

Pf: Let A, B and C be sets
Suppose $x \notin A$, $x \in B$ and $x \in C$
Then $x \notin (A \cap C)$, but $x \in (B \cap C)$

Therefore $(A \cap C) \subseteq (B \cap C)$ [Sofia did not end her sentences with periods.]

This behavior persisted for some time. For example, in a tutoring session in the middle of the semester, Sofia was trying to show the empty set is open in the relative topology on a subset Y of a space X. She wrote "Since $\emptyset \in \mathcal{U}$" and a little later said, "I'm stuck." The tutor then asked how in general one showed something was in $\{U \cap Y \mid U \in \mathcal{U}\}$. She answered, "By contradiction" without then attempting to implement or evaluate the suggestion.

Our intervention consisted of trying, during tutoring sessions, to prevent Sofia from enacting the "unreflective guess" schema by suggesting substitute actions. These included: draw a figure, look for inferences from the hypotheses, reflect on everything done so far, and do something else for a while. As the course ended, this intervention was beginning to show promise. For example on the in-class final examination Sofia proved that if f, g, and h are functions from a set to itself, f is one-to-one, and $f \circ g = f \circ h$, then $g = h$. Also on the take-home final, except for a small omission, she proved that the set of points on which two continuous functions between Hausdorff spaces agree is closed. This shows Sofia was able to complete the problem-oriented part of at least a few proofs by the end of the course, and suggests her "unreflective guess" behavioral schema was weakened.

7. CONCLUSION

This paper has discussed the nature of feelings, especially nonemotional cognitive feelings and treats them as part of affect. Behavioral schemas, or habits of mind, were introduced as a form of procedural knowledge that often automatically links situations to mental actions in the proving process. The way feelings can both arise from, and contribute to, the enactment of behavioral schemas was pointed out. An example of Mary and Dr. K was then provided to show how a feeling of appropriateness can be associated with, and strengthen, a behavioral schema that is often enacted in a student's proving process. Also our interpretation was given of a student's (Sofia's) persistent proving difficulty, an "unreflective guess" schema, based on a feeling of not having an idea of what to do next. This interpretation led to a beneficial intervention.

A further investigation of how nonemotional cognitive feelings and behavioral schemas relate to each other and to the proving process might well be beneficial in helping students learn to prove theorems.

Acknowledgement: We would like to thank Educational Advancement Foundation for its support in helping us develop and study the Modified Moore Method course mentioned above. Any ideas expressed in this paper are those of the authors and not of the Foundation.

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