Polygons, triangles and capes

Designing a one day team task for senior high school

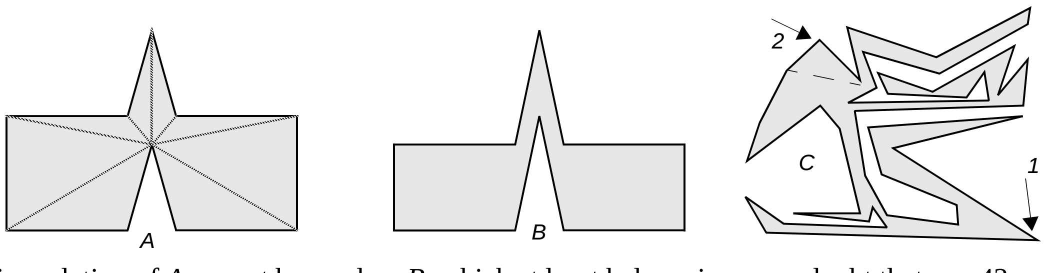
Aad Goddijn, Freudenthal Institute for Science and Mathematics Education

Introduction

The average mathematical competition for students, Junior High School to University, asks for solving well defined problems and favours non-standard tasks which require a considerable amount of mathematical ingenuity. Typically, contestants work on an individual basis; they often come to the competition as a team like in the IMO, but the real work is done in solitary mathematical silence.
Competitions for *teams* of students, where combatants of one team are supposed to cooperate, are a rare phenomenon in the field, but the Dutch Freudenthal Institute hosts two of them. The oldest one is the Math-A-lympiad, which has been running on a yearly basis from 1989 on and is meant for senior High School students preparing for academic studies in social or economical sciences. The second one is the so called Maths B-day, on which we focus in this contribution. It is meant for students preparing for science and technology.
Each year a special assignment for the Maths B-day is designed by a committee of three math teachers, two professional educational math designers of the Freudenthal Institute for Science and Mathematics Education, a professor in didactics of mathematics and a professional research mathematician. Students work on one task for one full day in teams of four. They produce (the same day!) a paper about it. The paper is part of the normal students' portfolio, but it is possible to send in the paper for the national Maths B-day competition.*
The design of this assignment has to take in consideration the special circumstances of the Math B-day. The design process will be discussed in this paper, preceded by a short exposition of its mathematical content, because it covers a not so well known subject.

Subject matter of the 2007 assignment: polygons, triangles and capes

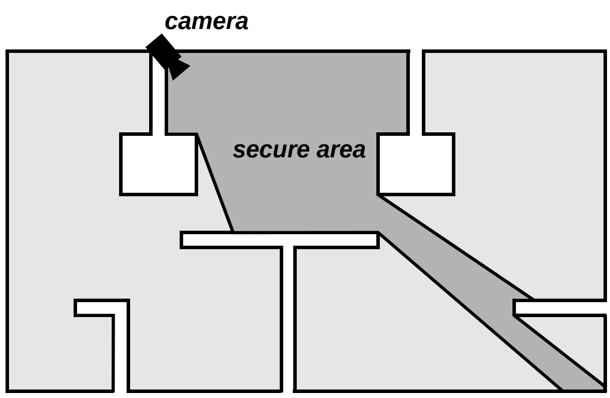
The assignment is available at the URL mentioned in the reference *Maths B-day design team (2007*). This is the text which students got on their desks, without any introduction about the content. They are supposed to work on there own.
In the assignment simple polygons are studied as the main subject; those are by definition closed polygons without self inter-sections. It is taken for granted that those polygons enclose a domain in the plane. Starting point is the search for (and proof of) a formula for the sum of the inner angles, expressed in the number n of vertices of the polygon. Several proofs are possible here, but a sequence of introductory task helps the students see that the usual proof of the well-known formula $(n - 2)\cdot 180°$ by dividing the polygon in n -2 triangles by diagonals from a vertex is at least debatable, by observing some examples like:



The triangulation of *A* cannot be used on *B*, which at least helps raise some doubt that any 43-gon *C* can be divided in 41 triangles at all.

* More information about the A-lympiad and the Maths B-day is available at the websites mentioned in the references.

The subject of polygon triangulation belongs to the field of computational geometry, a brand of mathematics were information science and geometry meet in a productive way and which has applications in many branches of science. Especially the art of finding better algorithms for large scale problems is still developing fast, even in the few years between O'Rourke (1987) and Berg, Kreveld, Overmars, (1997). Those are leading monographs in the field, which inspired the Maths B-day design team, together with a more educational text, Garton, (1997).
Our assignment focuses on the combinatorial aspects of the geometry, not on the order of algorithmic speed. Proof methods are indicated in the task, but not spelled out too much to give students the opportunity to build a real proof for the general triangulation-theorem. In a case like polygon C above it is easy to 'break off' successive triangles, but that is not yet general proof. First, because it presupposes the possibility of breaking up the polygon from the start on; second, because this approach doesn't indicate why for all very different polygons with *n* sides the same number of triangles is always required, namely *n* - 2. Also, there is no trivial general argument that a triangulation can always be made *without* introducing new vertices for the triangulation.
The problem of splitting of corners is studied in detail in the task. A vertex which can be trimmed off as a triangle with the neighbouring edges, is called 'a cape'. If a vertex is a cape, the neighbouring vertices are connected by a diagonal, that is: a vertex-vertex connecting segment which is inside the simple polygon. In example *C* vertex 2 is a cape, but vertex 1 is not. An important theorem: every simple polygon with four sides or more has at least two capes which are not neighbours. This theorem closes the gap in the initial 'proof' for the triangulation theorem, but is not trivial itself.
The assignments contains several open ended so called research problems, to give students opportunity to show their creativity. One of them is the study of the number of possible triangulations for convex polygons which leads naturally to the so-called Catalan numbers and a recursion formula to compute them. Another one is the charming *Art Gallery Theorem*, proved by Chvátal in 1973. It states that an area bounded by a *n*-sided simple polygon, as in a typical Art Gallery, can be viewed completely looked over by [*n*/3] cameras. ([*x*], the *floor of x,* is the largest integer not greater than *x*). The example below, taken from the task, can be guarded by less cameras than the general theorem predicts and of course one of the questions is for the minimal number of cameras.



Designing the task: overview of ideas

Task design for a classical mathematics competition will concentrate on finding and selecting the right mathematical problems. The situation is quite different for the A-lympiad and the Maths B-day.
A main question is how to handle the open-end character of the problem in relation to the vast differences in abilities of the students involved. It is a competition which the losers also have to learn something from! Typical for the current setting is the way the task design tries to supply some life lines for organization of the work and for clarity of defined concepts.
The mathematical complexity of the Maths B-day and the requirement that the task should lead teams of

students to a whole day of mathematical exploration and writing a paper needs an unusual design process. The competition character excludes try-outs on groups of students and subsequent rewriting almost totally, so no in-between testing of the design is possible.

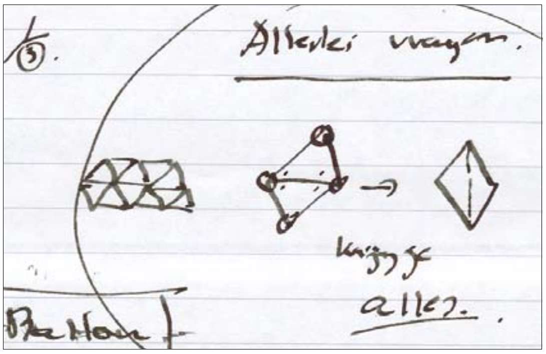
‘Guided reinvention’ is one of the flagships of the Realistic Mathematics Education approach, see for instance Freudenthal (1991). It is a general educational approach, visible in teaching and design both. In the current task, finding proofs and a recursion formula are mathematical tasks of a very high level. Without any guidance students in general can not be expected to come very far in this subject. We illuminate the guided reinvention aspect of the task design at the end.

Those main points will be evident when we first describe the design process as evolving form the committee’s mathematical fight with the problem itself.

Designing a NEW team task: the design teams mathematical activity as a starting point

The practical design process of the Maths B-day committee starts with an open confrontation of several initial ideas, proposed by different members of the team. One of the members of the committee proposed an interesting question in topology about triangulations of a sphere: how many different triangulations are possible with a fixed number of triangles?

In the committee this is seen as possible final research question of the assignment, where a lot of smaller problems have to be invented to help students explore this field in a beginners’ way, but the members of the committee typically take up the new question for exploration themselves, to experience in a hands-on way the mathematical and educational value of the idea. The number of possible 7-triangle coverings of the sphere was soon found. Euler’s theorem is a natural tool, but practical means of exploring were mentioned also, among others drawing on cheap party balloons.



A related idea concerns changing a given triangulation by swapping a diagonal in quadrilaterals formed by neighbouring triangles, as indicated in this sketch which escaped the wastebasket.

“*Krijg je alles*”: can you reach any triangulation with the same starting numbers of triangles by repeating swap-steps?

“*Allerlei vragen*”: lots of related questions are raised.

The initial enthusiasm for the problem was tempered by realizing the considerable difficulties young students could have already by acquiring a clear understanding of what ‘*equal*’ and ‘*different*’ for triangulations really mean in the topological context. But one of the members of the team offered to write an initial sketch of the assignment, as a starter for a 24-hour session.

For this sketch some further investigation was done. The swapping conjecture turned out to be true and the subject also turned out to be related to an application of geometry which is known in Dutch Senior High School: Voronoi-diagrams and Delauny-triangulations. But finding an elementary proof was not easy, an indication that it can be a far too high goal for the Maths B-day, where highly advanced tools are out of the question. Restricting to simple polygons and polyhedrons looked like a good option, but the general swap-theorem cannot be presented very well under this confinement of the geometrical freedom. Searching the internet with ‘polygons’ and ‘triangulation’ as keywords produced Garton (1997) as an inspiring source. Specially the metaphoric use of the terminology of ‘ears and mouths’ for what we called cape and cape-of-the-outside-area, suggested the field of simple polygons as a possible subject on its own. Elementary geometrical argumentations about capes, existence of diagonals, and dividing the polygon in parts were seen as possible fill-ins for the assignment as a whole.

During the 24-hour session away from office and city noise the dilemma was posed: pure combinatorial topology on the (topological) sphere or simple polygons. Two arguments tilted the scales to the poly-

gons: the theoretical/topological problems with the general approach and the rich possibilities of the simple polygons for simpler question to be usable in the assignment. Again the committee defined this way its fundamental setup of the Maths B-day assignment: there should be a large part of basic introductory material for all students, and there should be one or more open-ended research questions based on the introductory part for the teams who by their motivation and capacities are predestined to go for the competition.
The rest of the 24-session a lot of detailed proposals were made for possible use in the final task. Gradually attention shifted from the mathematics itself to the use of the results in the Maths B-day. A lot of those ideas can be found in the final design. For instance the stress on making sketches for all kind of polygons with special properties, the idea of finding counterexamples for apparently true statements about polygons, the idea of counting triangulations of convex polygons and the corresponding recursive relation of those numbers and the idea of exploring almost convex polygons (see later).
The final hours of the meeting showed new partial proofs for the Art Gallery and also the general outline of the proposed assignment.

The main point is that the design process of the committee started as a collective mathematical discovery trip in a field that was opened already a bit by some of the committee members. Doing and finding out the mathematics by themselves enabled the teachers involved to estimate what might be possible for their students, as the mathematical activity of the committee also sheds light on possible answers and approaches of students to the problems posed and on the possible obstacles they may experience. Finding the balance between introductory problems and advanced questions is only possible on the basis of insight in what students may run into during their work on the task.

The rough outline of the proposal already contained several of the so called research problems, their possible order in the task and decisions about what should be in the introductory part. No great changes in the basic outline were made later, but still it appeared that a lot of unforeseen details could turn out unnecessary to be obstacles.

The designers' mathematical activity as a starting point is mentioned by Hans Freudenthal in his introduction to *Five years IOWO* (1976) the institute he founded himself five years earlier:

> "IOWO is not a research institute; its members do not regard themselves as researchers but as producers of instruction, as engineers in the educational field, as curriculum developers. Engineering needs background research and can produce research as fall-out. Though both of them will be visible in the present account, its nucleus is our productive work, represented by a few specimens, and embodies our views on *mathematics* as a human activity and on curriculum development as a classroom activity, guided by curriculum developers, in close contact with all those interested in mathematics education."

The emphasis on mathematical activity may seem a bit stronger in the description of the Math B day design than in the quote of Freudenthal. But also with Freudenthal the emphasis is on *mathematics* [italics in the original publications] as a human activity, a background for engineering in mathematical education towards a design product.

Designing a team task for one whole day: provide help to organize the work

Experience with the earlier Maths B-days did reveal that the assignment must contain a good deal of basic introductory material for all students, and that there should be one or more open-ended research questions connected to the introductory material, for the 'specialists'.
In the introduction of the assignment a clear overview of the whole is given to the students, to help them organize themselves the work of the day:

– There is an A and a B paragraph; both basic if you want to go on to research tasks C and F, which are core pieces of the task as a whole.
– Paragraph D and E are side-steps, they are interesting on their own, but do not contribute to insight in the relation between C and F. They are extras.

Another piece of information which is important for planning the work day is the description of the final task, the paper to be delivered. It simply states:

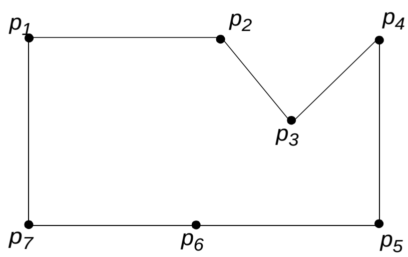
> *Write a report about the problem that is easy to read and can be understood by your fellow students even those who have never heard of simple polygons.*

The important thing is that it should be understandable for uninformed readers. The paper should clearly not be a list of answers to the guiding questions in the assignments. Students in general take this seriously. Of course they benefit from the guiding questions, but at least they try to build their answers into a ongoing mathematical text; so the requirements of the final paper help them think from an active overview of the subject, and not only from the details of the numbered questions, which is a far more passive approach.
Experience with the assignments of earlier years brought the committee to the combination of numbered questions and open ended tasks and also the readability-requirement of the final paper. Too open tasks turned out to provoke quite a bit of creativity, but a rather low level of mathematical organization.

Designing a team task for one whole day: the need for clear definitions

The 2007 assignment contains an element witch has probably been of considerable importance for the success of the day: a list of definitions. The need for this became clear when the final assignment was prepared and the concept of ‘angle of a polygon’ was introduced. A fragment of the assignment:

> **Angles of a simple polygon**
> *The sizes of the angles of a simple polygon are determined by the angles in the inside area*
>
> *On the right you see a polygon with 7 vertices.*
> *Straight angles, such as at vertex p_6, are also permitted!*



Straight angles at a vertex may be felt by the students as strange: why include them if the polygon line does not bend at them? The reason is that they can spontaneously arise by themselves when a cape is cut off, even if the original polygon did not have them. Students should be aware that when cutting off a cape, both neighbouring vertices survive the process. Allowing straight angles is just a practical move to prevent nasty case-distinction in proofs, theorems, and so on.

Another case is the definition for ‘convex polygon’. In the assignment it runs as follows:

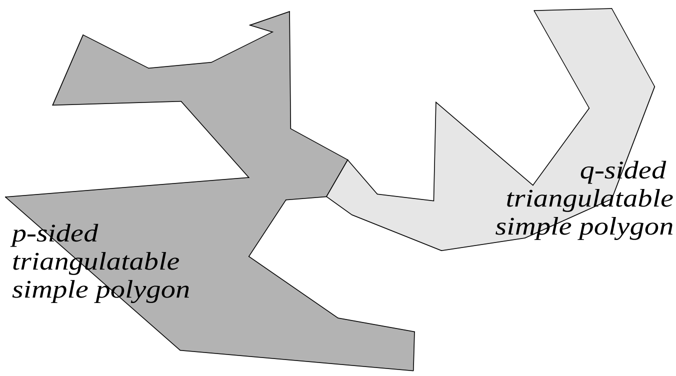
> *A simple polygon with only protruding angles is called a convex polygon*

Quite a clash with the usual definition for a convex shape!
This was done on purpose. Imagine triangulating polygons which are convex in the definition of the assignment: you may triangulate them by drawing all diagonals from one vertex of the polygon. Each choice of vertex leads to the same number of triangles. As counting triangulations of convex polygons leads to an interesting combinatorial approach, the given definition was chosen to prevent uninteresting distinctions of cases. The rectangle with an extra point spoils the combinatorics!

Of course defining concepts is an important part of doing mathematics on the level implied in the current problem situation. But leaving it to the students in a whole day task without content-guidance by the teacher would be very risky.
The main activities in the task have the mathematical character of argumentation and construction; it is in those aspects that mistakes were made and good ideas exposed. Misunderstandings about concepts played a minor role.

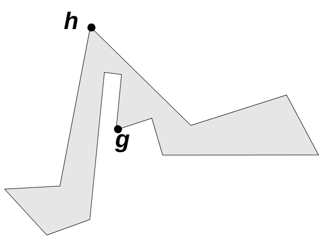
Designing for guided reinvention: the case of divide and conquer

Computational geometry as a mathematical discipline is concerned with constructing algorithms to solve problems like triangulate a given polygon and do at fast. In typical applications the polygons can be large and the methods to find the required object should be strictly described and should be guaranteed to deliver always the right results. In other words: algorithms needs generic proofs that they work in the first place and estimates of their 'speed' in the second place. In this Maths B-day task we do not pay attention to the *speed* of algorithms, but we certainly ask for *proofs*.
A typical proof pattern in the field is 'divide and conquer'. We did not expect all students to find the principle themselves, so we guided them very clearly to the principle and let them finish the proof as far as possible for them and we built into the task the possibility to use the method again in a different situation.
The idea is best illuminated by the proof of the triangulation theorem itself. In the picture below, taken from the assignment, a diagonal is drawn, dividing the simple polygon in two smaller polygons.



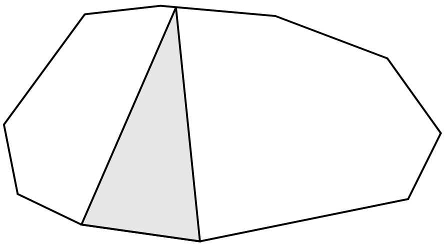
As the number of sides is n, we have $n = p + q - 2$. Students find their own way of proving from this that the polygon can be divided in $n - 2$ triangles, if the parts have the property of being dividable in (p - 2) and (q - 2) triangles respectively. Students' solutions differ in detail, in clarifying the assumptions for the parts for instance, or in explaining the found division as one step in the construction from the bottom up. This indicates that there is enough room left for students own constructions to fill in substantial parts of the proofs. The full proof is still quite subtle and many students make the same basic mistake in it, which again is possible because not the whole proof is a strictly guided tour.
As long as we haven't proved it, we are not allowed to use the fact that there *is* always at least one diagonal. The gap can be filled by the following statement, which must be proved:

If h is a protruding vertex but not a cape, then there is a diagonal from h.



Student solutions here typically state that if there is no cape at h, then the polygon will have a vertex inside the triangle formed by h and its neighbouring edges. This is true!
But most students conclude immediately that this vertex can be connected with h by a diagonal. That is not always true, as can be seen at vertex g in this picture.

In the case of the proof of the triangulation theorem students are guided and left to stand on their own feet at the same time. A principle is indicated (divide and conquer) and a strong hint is given. The main task, as can be seen in the assignment, is to bring together al the ingredients of the proof in a right and understandable way. So care is taken in the design that students are guided, but stay responsible themselves for what they do.
In a later stage of the assignment, part D, students work on systematically counting the number of different triangulations of given polygons. In the case of convex polygons there is a neat method to derive the numbers for larger polygons from the numbers for the smaller ones. In the assignment this is illustrated with a picture like this:



A triangulation which has the grey triangle as one of its members is found whenever we triangulate the neighbouring pentagon and hexagon. To get the total number of possibilities we have to multiply the numbers we found for pentagon and hexagon, and after this we have to sum over other choices of the grey triangle with the same base as the original triangle as a fixed edge and a variable top.
In the assignment again this plan is not written down, but there is just the question how many triangulations you may build, given the grey triangle is part of the triangulation. This way students bring in the combinatorial idea themselves and (re)invent the rest of the process, which is far more rewarding then working with too much guidance.

Concluding remarks
From a design point of view the Maths B-day task is an interesting case. The educational setting differs a lot from the regular one, which is reflected in goal, method and process of the design. We summarize our main points under those three headings.

Goals of the design for the Maths B-Day assignment:

1. An extended coherent task, interesting and varied enough to work on with a team of students for almost a day, instead of a sequence of disconnected task.
2. A task which is well accessible for the average student, and at the same time provocative for the high achievers.
3. A task which has a good balance between open ended problems with enough opportunity for the students own constructive contributions and responsibility, and guided reinvention, clearly defined questions and hints on the other side.
4. A task which asks for argumentation, proof and debate.

Design heuristics, the tricks of the trade:

5. The design of the Math B-day assignment requires an open eye for subjects which are off the beaten track of the school curriculum, but are well accessible from it. Knowledge of the history of mathematics and of old or modern applications of mathematics is helpful.
6. Reinventing part of the theory of the chosen problem field by the designers generates lots of ideas for possible use in the assignment.
7. Studying a meaningful problem (like the Art Gallery Theorem or the swapping of triangulations on the sphere) is an inspiring start for the design process.

The process of designing a team task for a day in mathematics

8. The design is best done by a team of designers. It is a try-out of the situation of working with a team on the problem where there can be no try-out with students, because of the competition character of the Math B-day. The team situation also helps to generate and immediately test new ideas about the problem and makes hidden obstacles visible. One should take considerable time to let this process evolve. A mixed team of teachers, educational designers and mathematicians who know and respect each other's capabilities is necessary.
9. Sources like internet, books, other professional mathematicians can best be accessed before the team meets for a longer meeting. During the meetings the creative interaction of the team should be the driving force.

Some of these points are of importance for all design activities in Mathematics Education, but 5 (off the beaten track), 8 and 9 (design by a mixed team), 6 (reinvention by the designer) are specially important for Math B-day approach.

References

Berg, M. de, Kreveld, M. J. & Overmars, M. (1997). *Computational geometry: algorithms and applications*. Berlin, Springer.

Freudenthal, H., Jansen, G.M. & Zweers, W., (eds.), (1976). *Five Years Iowo, On H. Freudenthal's Retirement from the Directorship of IOWO.* Educational Studies in Mathematics 7(3).

Freudenthal, H. (1991) *Revisiting mathematics education: China lectures*. Dordrecht, Kluwer Academic Publishers.

Garton, I. (1997). *Ear Cutting for Simple Polygons.* http://cgm.cs.mcgill.ca/~godfried/teaching/cg-projects/97/Ian/cutting_ears.html

Home page of the Math A-olympiad: *http://www.fi.uu.nl/alympiade/en* (in English)

Home page of the Maths B-day: *http://www.fi.uu.nl/wisbdag/* (For the English version of the assignment choose *2007* and look for *Engelse versi*e)

O'Rourke, J. (1987) *Art gallery theorems and algorithms*. Oxford, Oxford University Press