The teaching and learning of integral calculus from a historical perspective

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Conception of teachers on the teaching of mathematics

The teaching of science and mathematics in particular, even at the university level, it doesn´t give the students the opportunity to become familiar with the strategies characteristics of scientific work. Numerous studies in didactic of science relate to a distorted view of it, as one of the main obstacles to the renewal of its teaching.
Fernandez, Gil, Carrascosa, Cachapuz and Praia (2002) argue that, just as students have preconceptions that interfere with the acquisition of new knowledge, presumably also teachers have preconceptions about teaching, which could hinder the process of teaching.
In this regard, it's important to highlight the conception of teachers on how students learn and what is the nature of mathematical knowledge, in order to the proposals based on innovation of teaching can be properly carried into practice.
Fernandez et al.(2002) says that one of the reasons for the interest in the study of teaching conceptions about the nature of science, lies in the belief that these concepts include reductionism and deformation, which may be hindering the proper orientation of education. Specifically we refer to *a rigid, algorithmic, accurate and infallible vision of scientific activity*, from which the scientific method is presented as a set of steps to be followed mechanically, where the quantitative treatment and rigorous monitoring is prioritized against the potential for invention and creativity.
This is particularly evident with regard to the assessment: the obsessive concern to avoid ambiguity and to ensure the reliability of the assessments distort the nature of scientific work, essentially diffuse, uncertain, intuitive.
Very tied to that rigid vision, it´s mentioned the "*not problematic and not historical vision*". This position, which is considered closed and dogmatic, is the transmission of developed knowledge, without showing which were the problems that led to its construction, which have been their evolution, difficulties, disputes, or even much less, the limitations of current scientific knowledge or open prospects. One loses sight, as Bachelard says: "all knowledge is the answer to a question" (1938, cited in Fernandez et al. 2002: 480), to a problem, which makes it difficult to capture the rationality of the scientific process. It is very common for teachers of mathematics, by introducing new concepts, not make reference to the problems that lie at the root of the construction of such knowledge.
It is also worth noting *a simplistic interpretation of the development of mathematical knowledge*, where it´s common to introduce current accepted knowledge without showing how it was made, or refer to the frequent confrontations between rival theories, or to the complex processes of change.
Finally, we must take into account the existence of *an individualistic and elitist conception of scientific work* where the context of science is ignored. The scientific knowledge, mathematical discoveries, and so on, appear as the work of isolated geniuses, while the role of collective work of exchanges between teams is ignored. It contributes to this elitism, even more, with only operational lessons. There isn´t an effort to make mathematics accessible, starting with quality treatments and more meaningful, to show his human building character, in which confusion or mistakes are present, like those of the pupils themselves.
This teachers´ mindset is one of the determining factors in providing a teaching of mathematics usually based on finished facts, in which the student does not come to own the

notion of debate or dispute. It is usual to consider the construction of knowledge purely cumulative. That is why increasingly authors emphasize the need for teachers to value the importance of offering a vision of mathematics through the historical aspects that have influenced the construction of knowledge. In this sense, Garcia Cruz (1998) introduces the bachelardiano model (Bachelard, 1938), which is based on the idea of scientific change, within which there are three well-defined categories in the context of epistemology:

- Barriers epistemological: They are ingrained ways of thinking, old structures, both conceptual and methodological, impeding the progress of scientific knowledge.
- Breaks epistemological: In general terms, they are the ways in which scientific knowledge contradicts the ideas or beliefs that come from foremost primary knowledege, intuitive and common sense.
- The break that occurs between two different scientific concepts, both for a given knowledge to a specific methodology, it's also considered an epistemological break. Any break involves overcoming the corresponding barrier.
- Epistemological acts: They are the mechanisms by which epistemological obstacles are overcome and breaks with the old concepts are favored, causing corresponding changes and improving the scientific vision about reality. Within these mechanisms the use of the history of science plays a vital role, especially when attempting a reconstruction of the processes that has conditioned the progress of scientific knowledge.

Moreover, by virtue of the interest in improving science education has grown in recent times, which was a concern of teachers and education specialists, traditionally, has now involved researchers, who until recent were engaged in their respective scientific disciplines.(Paruelo, 2003)

This growing interest generates an increase in the quantity, quality and specificity in research on the subject, and it also facilitates the search for new strategies to address the teaching of science. In the area of new research groups specialized in the teaching of science, epistemology emerges as a suitable tool for the development of new strategies. Thus, the level at which the content and conceptual strategies in the classroom change under epistemological considerations, can enrich training in science if it's paid attention to certain difficulties that may arise in the implementation process. To take the epistemological contexts as reflections within a metacurriculum, presents difficulties in its implementation, because neither the teacher nor literature are appropriate. (Miguel, 2000).

If we focus our interest in the teaching process, and based on the concept of teaching transposition[1], it appears clearer the difference between what is called "the theory of scientists", "the theory taught by the teacher, and lastly , "the theory learned by the students." Paruelo (2003: 330) Each of these emerges as an adaptation of the above, where it's also present the adaptation of language, the type of mathematical tools used, and also, the degree of simplification on the applications, which varies depending on the level of education where the transfer is being done. In this regard, Paruelo (2003) believes that the distance will be higher in the initial levels of education, that is to say: the theory of scientists and the theory taught tend to resemble more and more as it progresses at the educational levels.

An introspection on the most common literature suggests that a wrong epistemological analysis is made in the process of didactic transposition,which leads to circularities or contradictions that create confusion in the students.

Epistemological barriers in the teaching of calculus

One of the didactic phenomena which is considered essential in the teaching of Mathematical

[1] Introduced by Chevallard (1991) for the purpose of appointing the transitional process ranging from "object to know" to "object of teaching".

Analysis, is the "*algebrización*", that is: the algebraic treatment of differential and integral calculation.

Artigue (in Contreras, 2000) expresses this fact in terms of an algebraic and reductionist approach of the calculation which is based on the algebraic operations with limits, differential and integral calculus, but it treats in a simplistic way the thinking and the specific techniques of analysis, such as the idea of instantaneous rate of change, or the study of the results of these reasons of change.

Contreras (2000) explains that when a teacher explains a particular mathematical concept and does not address, or does so on a superficial level, the typical problems of analysis, sliding into algorithmic postures easier to manage and evaluate, produces a genuine break of contract teaching, called by Brousseau "Topaze effect"(Brousseau, 1986)

Regarding the concept of integrate, we find it appropriate to make some considerations about the misconceptions in the students:

A) very often, students identify integrate with primitive. For these students, it's not involved any convergence process nor any geometric aspect in the calculation of the integral. It is therefore a purely algebraic process, more or less, and always self, so that a student can learn different methods of integration, and even apply them to calculate with some fluency (integral parts of rational functions, trigonometric integrals, etc.), and at the same time, not being able to solve the calculation of an area or ignore absolutely that it is a Riemann sum.

B) students identify definite integrals with the rule of Barrow, even where it's no proper to apply it. Such it's the case of essential discontinuity functions in the range considered. This type of answer seems to be representative of a deeper disconnection between the concept of integral and its particular image of that concept.

C) A third problem stems from the lack of association between the definite integral and the analysis of convergence. In general, when improper integrals are studied, the majority of students are very surprising that an integrate can be divergent.

D) Finally, it's not integrated the concept of area with the concept of definite integrate. It is likely that students have noticed that there is a relationship between both, the definite integrals and the area, but teachers don't work with the students on these issues, so that it remains a purely algebraic interpretation of the integral.

Students use the formal-algebraic context instead of visual-geometric one, simply, because they haven't integrated them. (Llorens and Santonja, 1997). A revision of precollege and college calculus textbooks, has provided us with illuminating patterns on the origin of the obstacles the students should face.

From a curriculum analysis, we can see that the sequence contained in the section of Integral Calculus is generally in the following order: 1) Calculation of primitives, 2) Methods of integration, 3) The definite integral: Barrow Rule, 4) Applications of integration: Calculation of areas and volumes.

This involves considering the integration primarily as the inverse operation of differentiation with the purpose of training students in the calculation of primitives. This is based on repeating a lot of exercises, demanding a substantial and progressive skill level, using tips and recipes that help to be more effective in obtaining results. In the analyzed texts, phrases such as :"... the trick that facilitates the process consists of multiplying the numerator and denominator by ... "or" ... use the replacement ... ", appear very frecuently.

The concept of integrate through history

The calculation of the area enclosed by a curve has been a problem from very old, very interesting for mathematicians. One of the major problems to solve was the comparison of figures curvilineas and rectilineas and it was Eudoxo (406 BC-355 BC), who gave the key to the solution. Mathematicians had earlier tried to circumscribe and inscribe rectilinear figures to the curvilinear figure and to multiply the number of sides indefinitely, thus rectilinear figures were increasingly approaching the curvilinear, but what they did not know was how to close reasoning, since they didn´t have the idea of limit and it would remain unknown for more than two millennia. According to Archimedes (286 BC-212 BC), it was Eudoxo who gave the axiom known today as Archimedes Axiom, which underlies the exhausción method, the Greek equivalent of the integral calculation (Boyer, 1992).

The Greeks made use of this property to prove theorems about the areas and volumes of curvilinear shapes. In the 12^{th} book of the Elements of Euclid it's demonstrated that the areas of two circles are interrelated as the square built on their diameters. This seems to have been the first theorem concerning the precise extent of curvilinear shapes, attributable to Eudoxo, which is the maximum contribution to mathematics made by the members of the Platonic Academy.

With the development of the calculation in the seventeenth century, it emerged the interest in the determination of the area bounded by the curve represented by a more general function and the x axis. At first, this interest was mainly geometric, which was to find ways to go beyond where the geniuses of the age had arrived.

Then mathematicians noticed that this area had, in many occasions practices, a very important meaning on the phenomenon that describes the function. The work being done by a force, the space that runs a mobile, spending power, the flow of water from a river, and many other problems, are given from the graph of a certain function. Indeed, the emergence of the functions instead of the curves as an object of study, allowed the gradual numeracy analysis growing and the consequent separation of geometry. The problem of integration was to identify a primitive function F(x) according to the derivative f(x). Thus, during the eighteenth century and part of the nineteenth, integral calculus was thought as the inverse of the differential.

During the nineteenth century there was a conceptual shift from what is a function: it was accepted the definition of a function of a real variable, as an arbitrary correspondence between real numbers. This situation led to the new approach on how to solve the integral of arbitrary functions that cuold not be expressed as equations. Then, a new concept of integrate arose. Cauchy presentes an integral definition as the limit of a sum, and a rigorous formulation of the fundamental theorem of calculation.

However, Pierre de Fermat, in the first half of the seventeenth century, was who provided the foundation for the development of differential calculus. Not available from the concept of limits, however, his method for determining maximum and minimum, followed a path in which modern writers have inspired and written books that are used in most universities where differential and integral calculus is teached. Fermat not only provided a method to find the tangent to the curves $y = x^m$ but also discovered a theorem concerning the enclosed area under these curves. His method allowed calculating the area under the curve $y = x^n$ between abscisa points x = 0 and x = a, dividing the interval in an infinite number of subintervals, taking abscisa points: *, aE, aE^2, aE^3*, ..., where E is a number smaller than 1. For each of these points, the value on "y" axis was calculated and the area under the curve through the sum of the areas of rectangles circumscribed, was approached. Areas of successive rectangles, starting with the largest, covering the point x = a, were given by the terms of an endless exponential progression: $a^n(a-aE)$, $a^nE^n\ (aE-aE^2)$, $a^nE^{2n}(aE^2-aE^3)$,...The sum of these terms is: $a^{n+1}(1-E)\ /\ 1-E^{n+1}$ *either:* $a^{n+1}\ /\ 1+E+E^2+...+E^n$. According to E tends to 1, that is to say: as rectangles are becoming ever closer, the sum of the areas of these rectangles is increasingly closer to the area under the curve, and taking E = 1 in the formula above, $a^{n+1}\ /\ n+1$ is obtained, allowing calculate the searched area under the curve $y=x^n$ from x = 0 to x = a. (Boyer, 1992). A similar demonstration was obtained for the case of fractional exponents p/q making a change of variables, where p/q = n. For negative values of n (except n = -1) Fermat used a similar procedure, except that, it considered E greater than 1 and it tended to 1 for higher values. The case n = -1 was resolved by Gregoire of St. Vincent in his Opus Geometricum. There, Gregoire proved that if we take, along the X axis, points from x = a, such that the ranges along X axis grow in geometric progression, and for those points we get the Y value for the hyperbola xy = 1, then the area under the curve between every two successive Y are the same, that is: according to the abscissa grows geometrically, the area under the curve grows arithmetically. Therefore, the equivalent of equality known today: $\int_a^b x^{-1}dx = \ln(b) - \ln(a)$, it was already familiar to Gregoire and his contemporaries.

It should be noted that both, Fermat as Gregoire, did not pay attention to the reverse character between derivation and integration. For Fermat, the calculation of the area under the curve x^n preceded the differentiation process, at least for integer values of n. In Gregoire´s work integral calculation also anticipated the differential calculation for the logarithmic function (Boyer, 1968).

A current estimate based on Fermat´s approaches

The failure of the students in understanding the concepts of calculation, more generally, and the definite integral, in particular, is one of the most worrisome problems in the learning of mathematical analysis in the first year of engineering, since this impedes understanding and solving many problems of implementation.

The way search of causality of this failure led us to deepening into the epistemological ground and raise the need for a change of conceptual approach.

We believe that a change in mathematical concepts taught should refer to the form of designing: a concepcional change rather than a conceptual one(White, 1994 cited in Pozo, 1999), that is a process and representation change by which pupils learn, in this case, the integral concept.

Focused on that goal, we proposed a teaching definite integral concept design, with the intention of producing epistemological ruptures, in response to how breaks in the history of mathematics occurred.

Since the process of construction of the area from a summation, which allows to add infinite quantities infinitely small, is difficult to understand by the students, it was used a software ad doc, built according to the needs of experiences and planned previously. It showed the geometric approximation, defaults and excess, between rectangles based increasingly small and the curvilinear area intended to be determined (see Figures 1 and 2).

The use of the computer was a valuable specific strategy with the goal to achieve meaningful learning. Several authors remark the methodological benefits of its incorporation: it fosters the development of the initiative in the pupils, it promotes a self rigorous and methodical work, pupil and teacher dialogue is becoming more open and rich, learning from the errors is stimulated, evaluation and monitoring of the work performed is easier, interdisciplinarity is promoted, development of activities is stimulated.

Similarly, and with similar tools, the teaching and learning of calculating the area between two curves, improper integrals, volumes of solid, lengths arc curve and average values of a function, were addressed. Faced with each new problem, the quantity to calculate, Q, was divided in a large number of small parties, an approximation of each fraction in the form $f(x_i)\,\Delta x$ *was obtained* and then, an approximate Q through a summation. Whenever possible, we obtained the exact value of Q through an integral, which makes sense when the limit of the sum is taken.

As for the problems of application to physics and engineering: work, pressure and hydrostatic force, calculating mass centers, we used the strategy of dividing the physical quantity in a large number of small parties, obtain an approximation of the fraction, add the results, take the limit and evaluate the resulting integrate.

The use of patterns emerged from the concept of integrate, constituted an invariant along the whole process of teaching and learning.

Conclusions

One of the most widespread general trends today, is the emphasis on the transmission of the thinking process of mathematics themselves, rather than the mere transfer of content. The math is, above all, know-how, it's a science in which the method predominates,clearly, on the content. That's why a great importance to studying these issues is given, largely adjacent to the cognitive psychology, which refer to the mental processes of problem solving.
With the purpose that students understand the fruitful interaction between reality and mathematics, we believe it is necessary to apply, on the one hand, the history of mathematics, enabling them to learn the process of emergence of mathematics at the time, and the other hand, to applications of mathematics, as they show the fertility and potency of this science. It is important that they know that math has done very similar to other sciences, by successive approximations, by experiments, by attempts, sometimes successful, other sterile until a more mature, but always perfectible, shape is reached. The historical reconstructions allow them to understand the changing nature of science.
Our ideal education should seek to reflect this deeply human nature of mathematics, thereby gaining in affordability, dynamism, interest and attraction. (Guzman, 1985)
Moreover it is necessary to update the content to be taught. In the situation of dizzying transformation of civilization in which we find ourselves, it is clear that the truly effective processes of thought, which don't become obsolete so rapidly, are the most valuable things that we can provide our students. In our scientific and intellectual world, so fast mutant, it's worth to summon up useful thinking process, than content, which quickly become what Whitehead called "lifeless ideas," ideas that form a heavy burden, which are not able to combine with others to form dynamic constellations, capable of addressing the problems of the present. The logical order is not necessarily the history order, and the didactic order doesn't match any of the two. But the teacher should know how things have happened, in order to:

- understand better the difficulties of generic man, of humanity, the development of the mathematical ideas, and through it, their own students ideas development,
- understand better the chronological sequence of ideas, the reasons and variations of the mathematics symphony,
- use this knowledge as a guide for his own pedagogy.

Knowledge of history provides a dynamic view of the mathematics evolution. That is where you can find the original ideas, in all its simplicity and originality, yet with a sense of adventure, which are often washed out in school texts. (Guzman, 1985 and 1986)
As rightly Toeplitz says: "With respect to all the basic themes of the infinitesimal calculus ... the mean value theorem, Taylor series ... never raises the question: Why so precisely? Or How was it obtained ? And yet all these issues have been, at some time, targets of an intense search, answers to burning questions ... If we returned to the origin of these ideas, they would lose their appearance of death and stuffed facts and would come back to take a fresh and vibrant life. " (Toeplitz, 1963)
Such a dynamic vision would train teachers and teacher educators for many interesting tasks in our educational work: the possibility of extrapolation into the future, immersion in the creative difficulties of the past, checking the tortuous invention roads, with the perception of ambiguity, darkness and initial confusion.
Unfortunately, both for the student who wishes to immerse himself in mathematical research, and for the teacher who wants to go about his applications or teaching, the history of mathematics is often totally absent from the university education in our country. We think it

would be extremely helpful if the various subjects we teach would be benefited from the perception of history, as we have tried to show in this brief paper, and that all of our students were provided with a comprehensive overview of the historical development of science that is going to occupy all his life.

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Figure 1
Approximation of the area under the curve C*os x* between –π/2 y π/2 , with 10 subdivisions

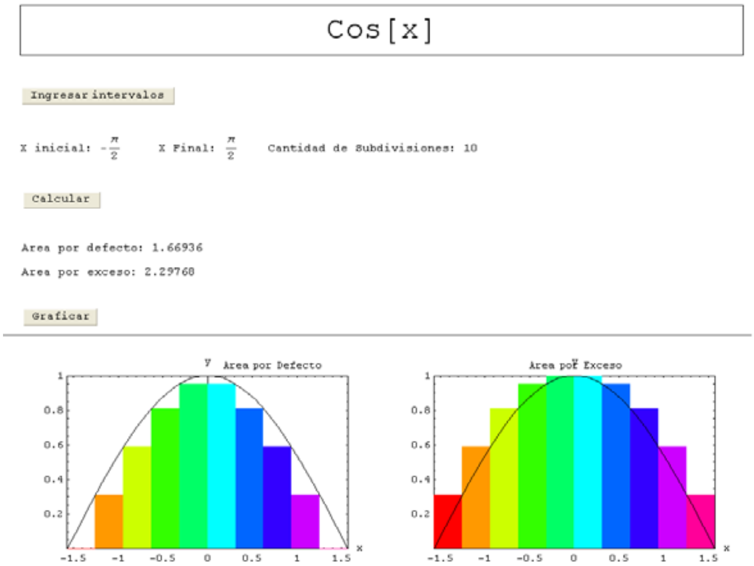

Figure 2
Approximation of the area under the curve C*os x* between –π/2 y π/2 , with 100 subdivisions