The development of Thales theorem throughout history

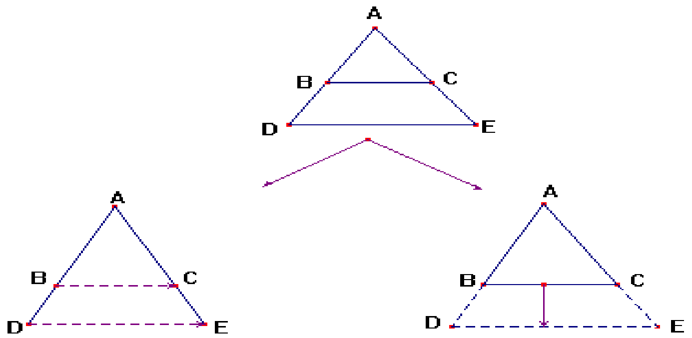
In the history, Thales theorem appeared with different forms. So, many statements formulated with distances, algebraic measures or vectors, related to a figure formed by two secants and parallel lines or to a figure related to two similar triangles express the same terminology. In this work, we try to explain the differences between these statements by characterizing the mathematical environments in which they evolve, which asks the question of the axiomatic followed in the current teaching, notably in the subjects around Thales theorem.

1. Thales theorem statements

We mean by Thales theorem the theorem which corresponds to Brousseau's classifications (1995) or Duperret's classifications (1995). Thus, we consider only plan statements and we remind that there are two essential approaches: "homothety" and "projection": the first is related to a triangle and allows passing from one triangle to another, the second can be characterized by one of the two figures: the figure "triangle" or the figure "parallel lines and secants", and allows passing from a straight line to another. To each one of these two approaches corresponds a vectorial writing:

- Figure "triangle"

The characteristic figure of Thales theorem in a triangle seems to be static whereas it hides two dynamic aspects which have been at its origin (Duperret, 1995):



"projection" Approach "homothety" approach

Fig 1

The "homothety" approach proposes ratio equalities related to two similar triangles: $\frac{AB}{AD} = \frac{AC}{AE} = \frac{BC}{DE}$.

In a vectorial form, it is defined by the relation: If $\overrightarrow{AB} = k\overrightarrow{AD}$ then $\overrightarrow{BC} = k\overrightarrow{DE}$

Brousseau (1995) distinguishes two cases in the "projection" approach according to whether, in each ratio, we choose the lengths of the segments of the same line or those of a line and their images on the other line (figures 2 and 3):

- "The conservation of abscissae on the secants"

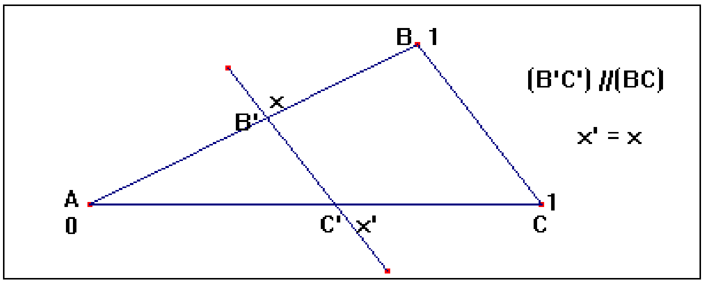


Fig 2 : "The conservation of abscissae on the secants"

We have $\frac{\overline{AB'}}{\overline{AB}} = \frac{\overline{AC'}}{\overline{AC}}$.

In a vectorial form: If $\overrightarrow{AB'} = k\overrightarrow{AB}$ then $\overrightarrow{AC'} = k\overrightarrow{AC}$.

- "The conservation of projection ratio"

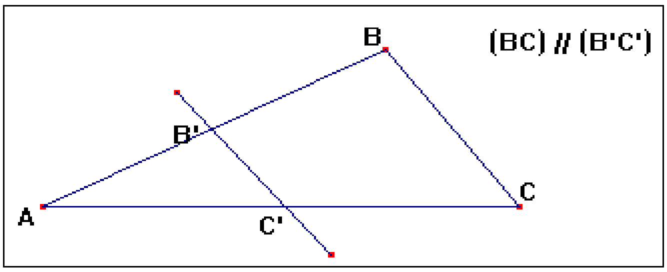


Fig 3: "The conservation of projection ratio"

We have: $\frac{AB}{AC} = \frac{AB'}{AC'}$

- Figure “parallel lines and secants”

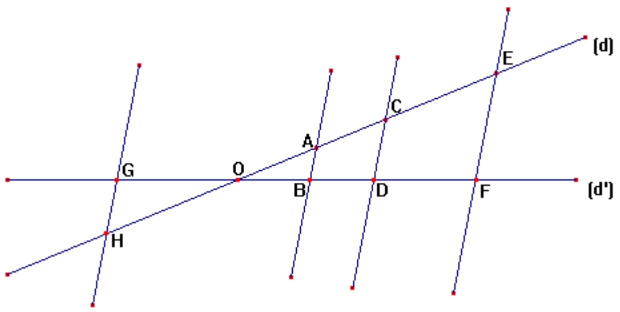


Fig 4

We can have forms of the type “The conservation of abscissae on the secants” or “The conservation of projection ratio”.

2. Thales theorem throughout history

We intend to analyze Thales theorem statements by using the various approaches which we have already stated, in order to deduce the mathematical environments in which this theorem evolves and the evolution of the various figures which characterize it. We choose to analyze particular books of geometry since Euclidean Elements and which are frequently used throughout history.

2.1 Euclidean Elements

Euclidean Elements provides us with the first statement in the history of Thales theorem. The proposition 2 of Book VI related to this statement treats the case of a triangle and a straight line parallel to one of its sides. Its demonstration is based on a preceding proposition (proposition 1 Book VI) which stipulates that “triangles and parallelograms having the same height are to one another as their bases”[1] and also on a cutting and a rebuilding of figures (annex). For Euclid, the method of areas has the advantage of avoiding the problem of the irrational because it does not require the construction of IR.

This statement uses only the segments which are placed on the sides of the triangle in the correspondence of “The conservation of abscissae on the secants”. In Euclidean Elements,

[1] Euclid. *The Thirteen Books of the Elements.* Translated with introduction and commentary by T. L. Heath. Second edition, Dover, New York, 1956, p.191

proposition 2 of Book VI does not seem to be an end in itself: it's its immediate application to the similar triangles which is generally used in the remaining Elements.

2.2 The Elements of Arnold (1667)

In the XVIIth century, Euclidean Elements were criticized by certain researchers as Arnold and Nicole (1995). The most linked criticisms to our topic deal with the respect of the real order of nature. In the demonstration of Thales theorem, Arnold rejects the using of areas by Euclid and prefers to use lines instead.

The first form in conformity with the recent statements of Thales theorem appears in a first corollary of the first theorem of Book X. This corollary stipulates that, if several lines, being differently inclined in the same parallel space, are all cut by parallel lines to this space, they are cut proportionally.

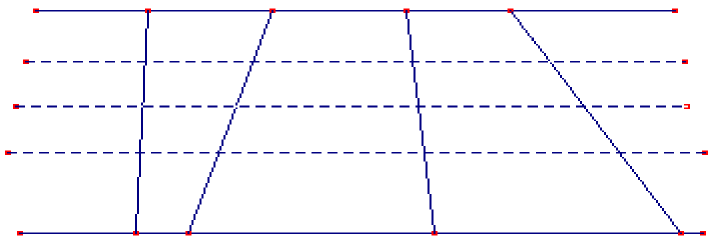


Fig 5

In Thales theorem applications, we find essentially the search for the 4th proportional, the division of a segment with given ratio. The type of Euclidean statement is found in Book XIII, and form only application of the principal statement of book X.

2.3 Legendre's Elements (1794)

Legendre's work marks a return to Euclid. Thales theorem is thus limited to the case of a triangle but we notice the rejection of surfaces' method. The proposition 13 of the book III indicates that:

Any parallel DE to one of the sides BC of a triangle ABC, divides the other sides AB and AC, in proportional parts.

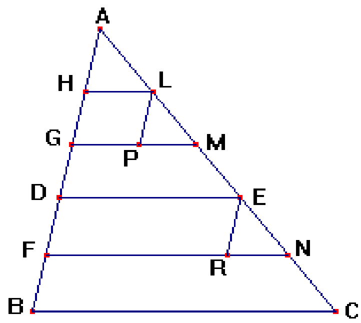


Fig 6

In Thales theorem applications, we find the properties of an angle bisectors in a triangle and the locus of the points for which the distances to two points B and C are in a given ratio $\frac{m}{n}$.

Then come the cases of similarity of the triangles and the applications to the similar polygons by breaking them up into similar triangles. The cohesion Thales - similar triangles is also obvious. Since their appearance, the similar triangles replace for Legendre, Thales theorem and find a rich field of applications.

In problems related to book III, the main applications that we find are: dividing a given straight line into as much equal parts as we want, or into proportional parts to given lines, finding a fourth proportional to given lines, and a proportional average between two given lines.

2.4 Hadamard (1928)

Hadamard is interested in the comparison of two figures and in the study of their correspondence by using two methods:

- the traditional method preferred by Euclid and which consists to apply the superposition principle.
- the “modern” method which calls the plan transformations. And then, orthogonal symmetry, rotation and translations are introduced since the first Book and cohabit with the traditional objects of geometry.

After a reminder of proportions and their proprieties in Book III, a statement of the type “The conservation of abscissae” is proposed. The associated figure deals with a series of parallels cut by two secants. The general case of this statement is prepared by a particular case where segments on the first line are equal.

The statement related to a triangle is immediately deduced from the first statement by passing a line by the vertex of a triangle in order to find the figure of the last statement.

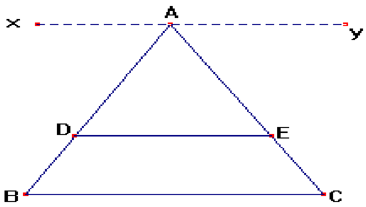


Fig 7

The immediate applications of Thales theorem deal with properties of the internal and external bisectors of an angle and the research of the locus of points for which the distances to two fixed points are in a given ratio. In these applications, tow results related respectively to the uniqueness of a point dividing internally or externally a segment in a given ratio, are of primary importance.

The cases of triangles similarity which follow replace the theorem which we are dealing with, and like with the preceding authors, henceforth they will be an important tool in the resolution of all Thales situations.

The concept of homothety, introduced with distances, consolidates for Hadamard, the dynamic aspect of geometrical figures and makes it possible to enlarge the similarity of the triangles to that of the figures and the polygons by breaking them up into triangles.

Let us note that since the 20th century, new conceptions of the geometry appeared. They benefit from linear algebra works and mark the abandonment of the inherited traditions of Euclid and his successors. With Choquet (1964), Dieudonné (1964) and others, geometry is based on concepts deduced from several centuries of research in mathematics: sets, order relations and equivalence, algebraic laws, vector spaces. With Choquet, Thales theorem is reduced to a unique form and makes it possible to establish the relation $\lambda(u+v) = \lambda u + \lambda v$ and then disappears immediately.

3. Mathematical environments around Thales theorem

From the preceding works, we can distinguish two mathematical environments which differ from the approach of Thales theorem that they use, from what makes it possible to arrive to this concept and from the corresponding field of applications. We focus particularly on the status of the figure[2] which passes, in these two environments, from a traditional form, closed and limited to a triangle, to a modern and dynamic form.

- ***The Euclidean mathematical environment***

In this environment, the main statement of Thales theorem is the Euclidean statement, which leads to "homothety approach".

The useful concepts and methods are essentially the case of triangles equality, proportionality, surfaces comparison and the method of cutting and rebuilding figures frequently used by Euclid. Many applications of Thales theorem appear in this environment: similar triangles, the cases of triangles similarity, trigonometric ratios, bisectors theorems, and metric relations which can lead to Pythagoras theorem. The problems which mostly characterize this environment are: the measurement of inaccessible objects, and the study of similar figures of the space.

In this mathematical environment, Thales figure is reduced to a triangle. The statement of the type "the conservation of abscissae" is not an end in itself: the cohesion between Thales theorem and similar triangles and more generally similar figures is a characteristic from this point of view.

- ***The mathematical environment of transformations:***

In the second half of the 19^{th} century, the concept of "figures movement" becomes fundamental and is inserted in geometry demonstrations marking then a rupture with the Euclidean traditions (Abdeljaouad, 2001). For this new point of view, a first remark consists in the dissociation of the cohesion Thales - similar triangles. Similar triangles are banned from the mathematical environment of Thales theorem. In this environment, the main statement of Thales theorem has the form of "the conservation of abscisea" or "conservation of projection ratio". The associated figure is formed by "parallel lines and secants". The useful concepts and methods are essentially: projection, projection ratio and graduated line. In the

[2] We exclude the intervention of Thales theorem in linear algebra and thus vectorial statements of this concept because this does not make in general an object of teaching

applications, the concepts and problems which rise from Thales theorem are reduced to homothety, and to only some traditional types of problems which are: segments division and the image of line graduation by projection.

The geometry based on transformations permit the movement of the figures and the transfer of properties between them. Chasles (1889) defines this point of view as a method which permits to find in a second figure the properties corresponding to a first one by the application of a transformation.

Conclusion

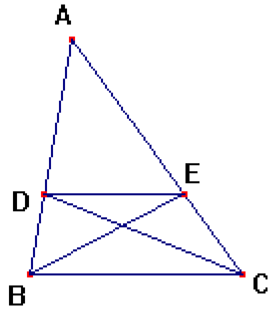
The historic works which we consulted show the existence of two different mathematical environments around Thales theorem, characterized by different approaches, figures and types of applications. They also prove that the figures “similar triangles” and “parallel lines and secants” are not equally representative of this concept. With the exception of Euclid for whom, the statement of Thales theorem is limited to a triangle, the authors start from one of these mathematical environments with the statements and the application types which characterize it to deduce the other mathematical environment.

While teaching, we think that all the statements of Thales theorem are worth using, and that each of them has to have enough time to be learned by the pupils and to be reorganized with regard to the others. We recommend to place each of these statements in the appropriate mathematical environment, which enhances the coherence of geometry subjects related with Thales theorem.

Annex

The proposition 2, Book VI of Euclidean Elements[3]

> *"If a straight line be drawn parallel to one on the sides of a triangle, it will cut the sides of the triangle proportionally; and, if the sides of a triangle be cut proportionally, the line joining the points of section will be parallel to the remaining side of the triangle ".*



<u>The proof:</u>

For let DE be drawn parallel to BC, one of the sides of the triangle ABC; I say that, as BD is to DA, so is CE to EA.

For let BE, CD be joined. Therefore the triangle BDE is equal to the triangle CDE; for they are on the same base DE and in the same parallels DE, BC. And the triangle ADE is another area. But equals have the same ratio to the same; therefore, as the triangle BDE is to the triangle ADE, so is the triangle CDE to the triangle ADE. But as the triangle BDE is to ADE, so is BD to DA; for, being under the same height, the perpendicular drawn from E to AB, they are to one another as their bases.

For the same reason also, as the triangle CDE is to ADE, so is CE to EA. Therefore also, as BD is to DA, so is CE to EA.

Again, let the sides AB, AC of the triangle ABC be cut proportionally, so that, as BD is to DA, so is CE to EA; and let DE be joined. I say that DE is parallel to BC.

For, with the same construction, since, as BD is to DA, so is CE to EA, but, as BD is to DA, so is the triangle BDE to the triangle ADE, and, as CE is to EA, so is the triangle CDE to the triangle ADE, therefore also, as the triangle BDE is to the triangle ADE, so is the triangle CDE to the triangle ADE. Therefore each of the triangles BDE, CDE has the same ratio to

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[3] Euclid. *The Thirteen Books of the Elements.* Translated with introduction and commentary by T. L. Heath. Second edition, Dover, New York, 1956, pp.194, 195

ADE. Therefore the triangle BDE is equal to the triangle CDE, and they are on the same base DE. But equal triangles which are on the same base are also in the same parallels. Therefore DE is parallel to BC. Therefore etc.

Bibliography

- Euclid. *The Thirteen Books of the Elements.* Translated with introduction and commentary by T. L. Heath. Second edition, Dover, New York, 1956.

- Abdeljaouad M (2001), Les triangles semblables…ces mal aimés, *Miftah al Hissab*, n° 96-97, Tunis.

- Arnauld A., Nicole P. (1995), La logique de Port-Royal, in « *Autour de Thalès* »,

inetr-Irem First Cycle Commission.

- Arnauld A (1667), *Nouveaux éléments de géométrie*, Charles Savreux, Paris.

- Arnauld. A., Nicole P., La logique de Port-Royal, in "*Autour de Thalès*", inetr-Irem First Cycle Commission.

- Brousseau G. (1995), "Promenade avec Thalès, de la Maternelle à l'Université", in "*Autour de Thalès*", inetr-Irem First Cycle Commission, p 87-124.

- Chasles M. (1889), *Aperçu historique des méthodes en géométrie*, ed. Gauthier- Villars, Paris.

- Choquet G. (1964), *L'enseignement de la géométrie*, Hermann, Paris.

- Dieudonné J. (1964), Algèbre linéaire et géométrie élémentaire, Hermann, Paris.

- Duperret J. C. (1995) "Pour un Thalès dynamique", in *"Autour de Thalès"*, inetr-Irem First Cycle Commission, p 125-144.

- Hadamard J. (1928), « Leçons de géométrie élémentaires », 10th edition, Paris.

- Legendre A.D. (1794), *Eléments de géométrie.* Firmin didot. Paris.