

The Huge Gap between Math Education and the front of Mathematics

(Preliminary Version)

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2000 Mathematics Subject Classification 97D20.

Abstract

The main purpose of this presentation is to show how mathematics education has been lagging behind the front of mathematics and science at least from the beginning of XX century on. We understand by mathematics education here, the mathematical cultural heritage we have been teaching and learning through the past generations from elementary school to the mathematical instruction given in college. We will analyze the effect caused on mathematics education by the fast growing of the mathematical knowledge in the past century. We try to understand and explain the great separation between math topics we are teaching, and today mathematics and science: mathematics and science on the headlines of the news. We look for new approaches to manage this gap, suggesting radical changes in the curriculum for elementary and high school mathematics.

Felix Klein and Hyman Bass will be mentioned here as milestone figures along the development of mathematical education in the past century. Both of them served as ICMI (*International Commission on Mathematical Instruction*) presidents; the former in 1908 and the latter finished his period in 2006.

RESUMEN

El propósito central de esta exposición es mostrar cómo la educación matemática ha quedado rezagada con relación a las matemáticas y a las ciencias en general. Entendemos aquí por educación matemática, el acervo cognitivo que el hombre adquiere a lo largo de su educación, entre el preescolar y la universidad, en lo relativo a las matemáticas. Se mostrará que los vertiginosos desarrollos matemáticos en la pasada centuria, no tuvieron mayor efecto en los desarrollos curriculares de las instituciones educativas. Se busca desentrañar las causas de la gran separación que existe entre lo que se enseña y lo que se investiga en matemáticas: entre lo que constituye un currículo en la enseñanza básica y lo que debería enseñarse en matemáticas para estrechar la brecha que separa lo que enseñamos y lo que es noticia en el mundo de las matemáticas.

Los nombres de Félix Klein, el gran matemático de la escuela de Gotinga y un gran educador, y de Hyman Bass, el pasado presidente de la *International Commission on Mathematical Instruction* (ICMI), serán reseñados aquí, como hitos históricos que representan dos épocas: una, de gran florecimiento de las matemáticas en Europa, y la época actual, caracterizada por su gran complejidad en lo que a educación se refiere.

1. – Introduction.

How could we prove that e and π are transcendental?, is one of the questions that Felix Klein answers, while teaching a course for high school teachers in Germany at the University of Göttingen around the year 1895¹. Göttingen would become “the Mecca of Mathematics” at the beginnings of XX century. It is remarkable that Klein was an intellectual too involved in mathematics education at all levels, a facet, no very well known of the German mathematician.

The proof of the transcendence² of number π was given by Ferdinand Lindemann in 1882. This achievement was a very high accomplishment, since, from the transcendence of π , follows the impossibility of the *quadrature of the circle*, one of the ancient Greek classical geometric problems.

After a few years from this achievement, we see Klein discussing these matters with high school teachers. It is interesting to know that at the end of XIX century, new discoveries within the field of mathematics would be known by mathematics teachers shortly after these results were discovered, and hence be known by teachers and students and consequently this knowledge be spread to become a part of social culture: the most ambitious aim, that any educational system looks for.

It should also be mentioned, that Leo Tolstoy, at the middle of XIX century, in his literature classic *War and Peace*, suggests searching the laws of history of human behavior, through the use of integral calculus, a subject, for this time, in process of development and consolidation in Germany and France³. Without entering in details, Tolstoy’s idea was to use infinitesimals to interpret historical episodes, and through them arrive to the governing laws of the – apparently chaotic – human behavior⁴. In this case, as in the Klein’s example, contemporary mathematics is used by Tolstoy, in his historic and literary speculations.

Nowadays these aspects look quite differently. Even though, having at hand, high technology, internet and enjoying better living conditions, we, most of math teachers, are disconnected from the frontiers of mathematics and science. If this is the case for math teachers, something worst, of course, happens to the ordinary citizen.

What did originate the huge separation between matters we teach and mathematics on the cover of science news?

¹ KLEIN, F. et al. *Famous Problems of Elementary Geometry and other Monographs*. Chelsea Publishing Company. New York. 2nd Edition. 1980.

² A real number is algebraic if it is a solution of a polynomial equation with integral coefficients. For instance, $\sqrt{2}$ is an algebraic number, since $\sqrt{2}$ is a solution of $x^2 - 2 = 0$. However, π is not in this category. Real numbers as π , are called transcendental or not algebraic.

³ See the interesting paper: AHEARN, S. T. *Tolstoy’s Integration Metaphor from War and Peace*. American Mathematical Monthly. August-September 2005.

⁴ A little more about this topic can be read in:

<http://www.matematicasyfilosofiaenelaula.info/articulos/Cronica%20XIX.pdf>

What can we do to reduce this separation?

Is this a local problem, or is indeed, a global one?

To make some reflections about these questions and analyze them is the aim of these notes.

Before trying the main issues, let's say a few words about two great mathematicians and math educators: Felix Klein and Hyman Bass. They are both taken into account here, as historical landmarks: the former representing a very creative epoch and the latter, at the contemporary scene, when math education is going through a serious crisis.

Felix Klein (1849-1925) was an influential mathematician into European intellectual circles at the end of XIX century and beginning of the past century. He was, indeed, a luminary of the *Göttingen mathematical school* and a passionate and creative mathematics educator.

Hyman Bass was honored the past year with *2006 National Medal of Science*. According to the selection committee, this award was given for: "his fundamental contributions to pure mathematics, especially in the creation of algebraic K-theory, his profound influence on mathematics education, and his service to the mathematics research and education communities. With his unique combination of gifts he has had enormous impact over the course of a half century"⁵. He is now at the University of Michigan and finished his period as president of the American Mathematical Society in 2006. One can see his involvement with math education through his many articles and as being the past president of ICMI (*International Commission of Mathematical Instruction*). ICMI started with the International Congress of Mathematicians, Rome 1908, as a proposal of the American mathematical historian: David E. Smith. By the way, the first president of this centenary institution was: Felix Klein. Bass' article, *Mathematics, Mathematicians and Mathematics Education*,⁶ show us, among other interesting things, his approach to some kind of hands on mathematical instruction, at the elementary level.

2. – Mathematics and Mathematics Education.

When one goes through the etymology study of the word "mathematics", we can find out, valuable aspects to be known. Jean Etienne Montucla (1725-1799), the first modern historian of mathematics, mentions that one of the first meanings for the word *mathematics* was general knowledge. Following Bochner⁷, the original meaning of the word *mathematics* was "something that has been learned or understood" from which he derives the meaning "knowledge acquirable by learning".

⁵ Additional information can be reached at: <http://www.ams.org/notices/200709/tx070901161p.pdf>

⁶ BASS, H. *Mathematics, Mathematicians and Mathematics Education*. Bulletin of the American Mathematical Society. Vol. 42, No. 4. October 2005.

⁷ BOCHNER, S. *The Role of Mathematics in the Rise of Science*. Princeton University Press. New Jersey. 1981. Pags. 24 and followings.

The original Greek word comes as *μαθηματικός* (English, mathematics, French, mathématiques, Spanish matemáticas). The change for “Mathematique”, in French and “Matemática” in Spanish comes from the time of the Bourbaki School, beginning at the thirties in the past century. We’ll speak about this influential mathematical school later on.

From above it follows that, from its own origins, mathematics is teaching and consequently, also learning. This means: *mathematics education is connatural to mathematics itself*.

There are lots and lots of literature about **how** to teach mathematics at all levels. However, it does not occur the same relating to **what kind of mathematics** we have to teach at school and college levels. The central thesis here lays on questioning the **mathematics contents** of our actual curricula. We want to point out, the urgency of putting under a new context the part of mathematics we are teaching as a part of our cultural legacy, i. e. mathematics we are transmitting to new generations, for whom, technology and way of life have changed so radically. These new generations are confronting realities quite different to those we had to face in the past. To understand the complexities of new technology and the intricacies of the information age, we need mathematics much different as that the students learned more than one hundred years ago.

When we take a look, for instance, to algebra texts used in high schools and colleges at the beginning of XX century, say, Chrystal’s Algebra⁸, we observe that there are no substantial changes in contents, comparing with current textbooks in this subject. It seams – instead – mathematics contents are degrading with time. We are probably teaching less mathematics now, than a century ago.

We are creating a great volume of mathematics nowadays, as it can be verified through the web and through the many mathematical journals all around the world. This great amount of mathematics, of course, cannot go directly to the classroom. However, after some decantation, some parts of it would be suitable to be taught either at high school or at college level. There are no enough pedagogical or scientific reasons for not teaching at high school, for example, either Riemannian geometry, or non Euclidean geometries, after almost two centuries since these topics were discovered. Why not teaching the Simplex Method at high school, knowing that mathematics involved in it is not so sophisticated? This practical method was created by George Dantzig (1914-2006) for solving problems in linear programming in 1947. And a last example, why not to talk about Poincaré’s Conjecture, solved by Grigori Perelman a few years ago? This famous conjecture could be a departure’s point and a motivation for introducing new and fascinating areas of mathematics, among them, topology, differential geometry and theory of manifolds.

3. – The boom of Mathematics in XIX Century Germany. A case study.

⁸ CHRYSTAL, G. *Algebra. An Elementary Text-Book for the Higher Classes of Secondary Schools and for Colleges. Two Volumes*. 7th. Edition. Chelsea. New York. 1964

Many cultured people are asking themselves, why at the same country or around the same cultural heritage, at nineteenth century did flourish so many high ranked mathematicians? There are no simple answers, of course, but only social and political explanations. At the beginning of that century an education revolution did occur which changed forever the educational tradition in Germany.

After having been defeated by France at the beginning of XIX century, Frederick William III, King of Prussia, was convinced that the most important surviving value from the war had been the young blood and in order to assure the future of his country a good education, would be a priority. His great hit was to assign William von Humboldt (1767-1835), Alexander von Humboldt's elder brother as education minister. William was a highly respected humanist of his time⁹. Both brothers studied at Göttingen, William was a philosopher, an anthropologist and a linguist, while Alexander was universally known as a naturalist. William von Humboldt took under his responsibility the reorganization of Prussia's public education system. Among important matters to remark, are the creation of a system of normal schools for teaching teachers of elementary school and the creation of a whole education system of grammar and high schools. Notably was the appearance of six hours of mathematics per week, along the years of basic instruction.

At the time of von Humboldt's administration was the University of Berlin created and it was setting out a clear philosophy for university education with emphasis in *freedom and universality*. Its original name was *Friederich Wilhelm Universität*. Now it bears the name: *von Humboldt University*, making a remembrance to his founder William von Humboldt, who was the creator of a new concept of university. It was at this university, where the term "Ph. D." appears for the first time, designing the highest academic title in sciences and humanities. This term comes from the Latin words *Philosophiae Doctor*, with a meaning like: "Philosophy's Teacher of Teachers" or "Learned Professional in Philosophy".

Carl Jacobi, Lejeune Dirichlet, Hermann Grassmann, Ernst Kummer, Karl Weierstrass, were among the mathematics crop of the new era. Students or under the influence of the already mentioned, appeared Heine, Kronecker, Riemann, Dedekind, Hankel, Cantor and the list will also bear famous names as, Klein, Frege, Lindemann, Hilbert, Hausdorff, Zermelo and many more.

University of Göttingen played an important role in this educational changing process, because of its efforts in the formation of teachers and researchers around this time. This tradition was kept alive until reaching its pinnacle at times of Felix Klein, David Hilbert and other distinguished mathematicians at the beginnings of XX century. Modern universities try to emulate the level and quality of the German universities at the time when the enlightenment reached Germany at the dawn of XIX century. Enlightenment was a socio-cultural movement around some European countries culminating with the French revolution in 1789.

⁹ A general description of the contributions to theoretical and practical education by Wilhelm von Humboldt can be read at: *WILHELM VON HUMBOLDT (1767-1835)*. Prospects: The Quarterly Review of Comparative Education. Paris, UNESCO: International Bureau of Education, vol. XXIII, no. 3/4, 1993, p. 613–23.

4. – The huge gap between what we teach and the front of mathematics.

The end of XIX century and the beginning of XX century was a time of great dynamism in creation of mathematics. This thrust came from specific intellectual currents mainly from France, Germany and Great Britain. Around this time arose some, now important areas within mathematics such as: topology, modern algebra, algebraic and analytic number theory, functional analysis, measure theory, complex analysis, mathematical logic, mathematical analysis in its abstract branches, probability theory and many other branches from the main fields. At the end of XX century the number of important branches of mathematics can be counted between sixty and seventy.

As a consequence of this fast mathematics development, mathematics education, mainly at the primary and secondary level, enclosed into the traditional courses of Arithmetic, Elementary Algebra, Euclidean Geometry and Infinitesimal Calculus, was unable to assimilate those fast changes and remained at the back, almost as a fossilized matter.

It seems that we are teaching mathematics for handling a kind of mathematics and scientific problems from the past or for a culture different to ours own. Problems new generations are facing are very different compared to those in Euclid time or those when the renaissance algebra appeared or yet, more complex than those treated for mathematicians during the European enlightenment around XVIII century. With the limited resources of elementary arithmetic and algebra, with Euclidean geometry and Newton-Leibniz infinitesimal calculus we teach at high school, it is not possible to understand the problems arising in our time, such as Poincaré Conjecture, on the news in 2006, or the mapping of E_8 , an outstanding result proved in 2007. To understand the meaning and importance of these results we need to know at least the elementary language of topology and differential geometry in the first case and some knowledge of Lie Algebras in the second. Subjects as those mentioned above, are comprehensible for ordinary people if some basics on those fields are introduced at high school curriculum.

The main thesis in this exposition is that we *must* change the mathematical contents in the elementary and high school curricula in such a way that the new students have the opportunity to be in touch with new subjects as mentioned above and less newer ones such as Riemannian geometry, Fourier series and the like. These subjects, among others, must be a part of the mathematics culture of the future citizen, if we want mathematics to continue playing an important role as has been the case in education during the past two millennia.

Vast parts of modern mathematics remain unknown for mathematics teachers and also for his students at high school and also, in many cases, by instructors at the college level. For instance, important and profound results, as *Gödel's Incompleteness Theorems* remain without being studied or commented in the basic formation of the modern citizen. I think much of set theory and mathematical logic, we are teaching at basic levels is useless. With this incipient knowledge of set theory and mathematical logic we cannot make mathematics more rigorous or sounder. It would be advisable, of course, to teach a well founded mathematical logic course to prospective

mathematics teachers, in order to be in touch with this very expanding and important branch of mathematics.

Most teachers at basic levels are not aware of the natural curiosity of their students related towards modern technology. It could be interesting to make our students understand the mathematical principles behind technology they are faced with on a daily basis, such as digital recording, banking encryption or cell phone communication. These areas within mathematics, albeit advanced, can be introduced in an informal way along either, the mathematical courses in high school, or in the mathematics courses for prospective high school teachers. Why not, for instance, introduce Fourier series, *wavelets*, and mathematics supporting modern technology, as early as advisable?¹⁰

5. – Abridging the gap between the front of math and the mathematics we teach.

The problem of mathematical education is enclosed within the more general context of the educational system we are immersed in. Mathematics education has a high specific weight in the cultural formation of man as a social being and for that reason we, as mathematics teachers, have to be very much concerned about its improving and put it permanently on a modern basis. Mathematics is engaged to education ever since the beginnings of western civilization, starting with Egyptian and Babylonian civilizations and following with the Greek culture, to which we are so much indebted.

Doing some review on the history of philosophy and mathematics, one finds out that these two areas of knowledge have been paired to education. At the beginnings of the Greek culture mathematics and philosophy were cultivated simultaneously. Zeno, Plato and Aristotle are names of philosophers who also made contributions to mathematics and education. More recently, Descartes and Leibniz were distinguished philosophers and mathematicians. Nowadays mathematicians are contributing with philosophy's growing, as it is shown in the honored list of the *Schock Prize* in Logic and Philosophy from the Royal Swedish Academy of Sciences. The mathematicians Solomon Feferman (2003) and Jaakko Hintikka (2005) were the last honored with this famous prize equivalent to a Nobel.¹¹

To abridge the gap we are facing, we need at first to recognize the problem in its real magnitude and then to look for intelligent alternatives to change contents of the basic mathematical curriculum, using new technologies to simplify the teaching of certain arithmetic and algebraic routines, which, in past times, we had to learn by heart. Much about the routines we learn in the elementary school could be shortened through the use of modern technology. Instead of spending much time teaching those routines, we can teach a little more of number theory and ways of applying mathematics to solve immediate problems (no ideal ones) facing children according their age. We have to create new courses for embodying old ones trying to keep the most

¹⁰ Something else about this, could be found at::

<http://www.matematicasyfilosofiaenelaula.info/articulos/Cronica%20XIX.pdf>

¹¹ Information relating these prizes can be found at:

http://www.kva.se/KVA_Root/eng/awards/international/schock/index.asp

valuable of the mathematical culture. In this process we can follow the ideas proposed by professor Bass¹² and colleagues from University of Michigan relating to a *Mathematical Knowledge for Teaching* (MKT).

MKT has four main components: (1) *Common mathematical Knowledge*. (2) *Specialized mathematical knowledge*. (3) *Knowledge of mathematics and students*. (4) *Knowledge of mathematics and teaching*. MKT let us, as mathematics professors, to furnish adequately the contents and vision of our mathematical courses accordingly, if these courses are: for basic schools (expected to be known by any educated adult), for teaching teachers (where some pedagogical training can be combined with mathematics), for colleges of liberal arts and sciences and courses for expected mathematics professionals. In some measure, each mentioned course would have certain amount from the components (1) to (4).

In Greek times there was a clear distinction between *logistics* and *arithmetic*. The former was the practical art of computing with numbers, while the latter was the study of the properties and relationships connecting numbers. Logistics was considered a kind of discipline of low profile and Greek mathematicians did not contribute much to its development. However, arithmetic, understood today as number theory, was a very important subject. Pythagoreans and the like discovered marvelous properties of numbers and much of their discoveries were compiled by Euclid in the Elements.

At elementary school teaching we would try new and revolutionary methodologies to introduce counting numbers arithmetic through no conventional ways. One way to do it could be using the most basic and elementary way to grasp the notion of quantity: *all or nothing*. Using this idea it is possible to arrive to binary system and doing arithmetic within it. The following could be an approach to introduce the binary system.

Child captures the idea of counting numbers since his earliest infancy – even before he knows figure representation for them – just using number words. After some time he or she discovers the possibility to classify counting numbers as odd and even through the ludics of very simple rounds and joyful plays. This way to see counting numbers, let children, learn the multiplication table for number 2, not just from 1 to 10, but indefinitely, 2, 4, 6, 8, 10, 12, With the help of some practices they can get the arrangement 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, ... , the generalized number 3 table. Continuing this way, children arrive to a generalization of multiplication tables for the numbers 2, 3, 4, 5, From here, they possibly get the idea of residual classes and without mental effort arrive to modular arithmetic (arithmetic induced by a congruence relation, the same one studied by Gauss in *Disquisitiones Arithmeticae* from 1801) and understanding $\{\mathbf{Z}/n\mathbf{Z}\}$, as a partition of integers and opening naturally a window to look inside abstract algebra. Examples of modular arithmetic abound and are very close to children reaching, for instance, the lecture of the clock hour, the musical scale, etc.

Now, that children have the idea of modular arithmetic, it is no difficult to engage them in looking for representation of numbers. Remember, we arrive first to the idea of counting numbers, not to the figures for those counting numbers. The figures are dead marks, no the

¹² Bass, H. *Op. Cit.* Pag. 429.

numbers themselves. How to represent those numbers is now the question. The more immediate way to get number symbols could be through the use of the most primitive number concepts, say **0**, and **1** (let's call them **void** and **full**, **nothing** and **all**). From here we'll be able to get labels for the counting numbers, 0, 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111, If children capture the way how to label numbers with these basic symbols, we are in position to introduce the concept of a base to represent or making number marks to identify numbers as such.

Under the basis of this elementary fashion of labeling numbers is the number 2. This means that just with two "digits" we can represent all natural numbers, including zero. From here on it will not be difficult to introduce the "value" – let's call it the floor – in the natural sequence of the numbers. This means, in the binary case, that 0 and 1 are at the ground level, while at first floor, will be 10 (one zero) and 11 (one one), at second floor 100, 101, 110 and 111 and for each floor up, numbers keep growing and growing. The important thing to remark here is that this growing is not capricious but follows a very simple rule: from a floor to the following the numbers appearing are double as in the immediate inferior level. In other words, in the first floor there are 2 numbers (2 raised to the first power), at the second floor, 2 times 2 = 4 (2 raised to the second power) and so on.

From here children are in conditions to give a jump to powers of the form 2^n , where n means the floor or level to which numbers belong. Note that, after addition and multiplication, powers are the following easiest algebraic operation. When representing numbers from the natural sequence, we can show children how these numbers are sums of numbers taken from the same floor and from lower floors. For example 1111 is the sum of 1000 (third floor), plus 100 (second floor), plus 10 (first floor), plus 1 (ground floor). If we convince our students that the appearance of n zeros after 1 means 2^n , we have gained a great battle, since from this point on we can introduce the representation:

$$1111 = 1 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$$

We can see the appearance of the polynomial $1 \times x^3 + 1 \times x^2 + 1 \times x^1 + 1 \times x^0$ (when $x = 2$).

Here "x" means, for just now, the digit association to the corresponding floor. After some practice easily you can catch for them the idea of polynomials, the easiest of all algebraic entities.

From this stage of the learning process of number representations you can easily go to the algorithms for addition and with some patience you arrive also to the multiplication algorithm. For the case of addition of two numbers (here we identify numbers with number representation), we start with the most simple sum table we ever can learn, namely:

+	0	1
0	0	1
1	1	10

This table only means that if you want to add numbers in the ground floor all you have to remember is that $1 + 1$ is 10 ; the remaining entries look obvious. It means that adding two ones is the same as to climb up to the following floor. With this in mind addition is nothing else than polynomial addition. Let us practice with a simple example. Suppose we want add $1111 + 1010$. Using the above representation and taking representation for 1010 :

$$\begin{aligned} 1111 &= 1 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 \\ 1010 &= 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 \end{aligned}$$

We find:

$$1111 + 1010 = (1 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0) + (1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 0 \times 2^0).$$

Of course, we have to say something about the use of parenthesis and the freedom we have to move and associate parts of the representation in such way that we put together elements from the same floors. So the sum can be arranged as

$$1111 + 1010 = (1 \times 2^3 + 1 \times 2^3) + (1 \times 2^2 + 0 \times 2^2) + (1 \times 2^1 + 1 \times 2^1) + (1 \times 2^0 + 0 \times 2^0).$$

At this stage of the process, we see four additions of homogeneous numbers (each pair at the same floor). Now we can use the basic adding table as follows $1 + 1 = 10$, $1 + 0 = 1$; $1 + 1 = 10$, $1 + 0 = 1$ and so we get:

$$1111 + 1010 = (10 \times 2^3) + (1 \times 2^2) + (10 \times 2^1) + (1 \times 2^0)$$

The meaning of 10×2^3 , according the rule of the jump to the upper floor, is 1×2^4 and at the third floor is left zero. Also $10 \times 2^1 = 1 \times 2^2$ and zero at level one. Changing the values, above, we get:

$$\begin{aligned} 1111 + 1010 &= (1 \times 2^4) + (0 \times 2^3) + (1 \times 2^2) + (1 \times 2^2) + (0 \times 2^1) + (1 \times 2^0) = (1 \times 2^4) + \\ &(0 \times 2^3) + (10 \times 2^2) + (0 \times 2^1) + (1 \times 2^0) = (1 \times 2^4) + (0 \times 2^3) + (10 \times 2^2) + (0 \times 2^1) + (1 \times 2^0) \end{aligned}$$

Repeating the use of the rule of the jump to the upper floor, $10 \times 2^2 = 1 \times 2^3 + 0 \times 2^2$. With this changes we get

$$1111 + 1010 = 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 11001.$$

But, what's remarkable about this naïve way to do arithmetic? The important thing here is that we are giving a foundation for the sum algorithm,

$$\begin{array}{r} 1111 \\ +1010 \\ \hline 11001 \end{array}$$

So, we are making more arithmetic here than logistics. We are also giving a reason for the addition algorithm, different to the traditional method of teaching algorithms by hearth without any justification behind it.

This process can be repeated with 3, 4, etc. When we arrive to 10 we are in the Al-Khwarizmi arithmetic, we have been teaching since middle ages.

Then, comes the question, why do not teach these elementary things to our children at grammar schools? I think the answer is because of the power of “tradition”; for the same reason that we are teaching Euclidean geometry and not a realistic type of geometry as Riemannian geometry.

The above arithmetic digression tries to reinforce the idea that we should may teach at elementary school more arithmetic and less logistics and show how it is possible to introduce “advanced” concepts like polynomials, modular arithmetic and consequently the basics of ring theory at the beginning of mathematics teaching. And most important than all, we are teaching the binary system to our children now, to expect one thousand more high performance in some years, than the results that modern computer gave us after fifty years we taught it, this same binary system.

William Thurston, Deborah Ball and Hyman Bass have suggested making some kind of “compression” of mathematics in such way that we can manage the great amounts of new mathematics to be comprehensible to human mind. A similar process of *compression* we have to follow to make possible teaching math that lets students to a comprehension of advanced mathematics.

6 – Some past experiences: The Bourbaki School.

One of the reasons for the creation of the Bourbaki School was the desire of some French mathematicians to reduce the distance between the mathematics taught at schools and colleges in the 1930s and the mathematics that research mathematicians were creating at the end of nineteen century and going through the beginning of twentieth century. André Weil writes in his memories¹³ how, his worries about this great gap, was shared by his colleagues of the Ecole Normale in Paris: Henri Cartan, Jean Dieudonné, Charles Pisot, Claude Chevalley and Jean Delsarte. This group of math professors was at the starting kernel of the Bourbaki School. This school would have, with passing of the years, a strong influence in western mathematical circles.

The first aim of the group was to write mathematical textbooks suitable for college level in such a way that they would be the basis to give the jump toward the advanced mathematics developed until that period in Europe and elsewhere. The proposed books must include the new Fraenkel-Zermelo set theory, in part axiomatized for them and also some fundamental principles that let to introduce the most modern tendencies of mathematics. Texts as Goursat Analysis in that time looked old fashioned and no apt for the exigent students of the Ecole Normale. The Bourbaki School was interested in introducing as soon as possible topics from vector analysis, created by Hermann Grassman at middle of XIX century. Only with the introduction of Hilbert spaces at the

¹³ WEIL, A. *The Apprenticeship of a Mathematician*. Birkhäuser Verlag. Basel-Boston-Berlin. 1991.

beginning of XX century, vector analysis was at the focus of mathematical research. An interesting story about the Grassman discovery is described in *A Metaphor for Mathematics Education*¹⁴.

Besides closing the gap between mathematics education and the front of mathematics, the Bourbaki School tried, as Armand Borel says¹⁵, "...they wanted to adopt a more precise, rigorous style of exposition than had been traditionally used in France, so they decided to start from scratch...". This rigor in the presentation of all subjects was its own characteristic, where the concepts of structure and homomorphism were always present along the volumes of the most known of their works: *The Elements of Mathematics*. Its influence spread around the world, not only in mathematics, but also in philosophy, psychology, linguistics and of course, in mathematics education, with much questioned consequences in several countries, where the *modern or new math*, inspired in Bourbaki, took all levels of mathematics education.

New Math in USA and Latin America. In Colombia and Latin America the new math arrived in around the 1960's with its great charge of non digested logic and lots of set theory at such extreme that set theory was introduced at kinder level and following up till the college level. In the United States, SMSG (*School Mathematics Study Group*), had in the Bourbaki philosophy, the model to follow with very critical consequences, specially at basic school.

The so called New Math brought, as its main ingredient, the axiomatic method, enclosed in a formalistic frame, which was imposed from the most elementary stages of mathematics education. The consequences were catastrophic at such a point that a broad movement of counter reaction began to operate at universities where mathematicians fought against this type of mathematics taught at school. Among the high figures in these math wars, was Morris Kline, from New York University, who wrote *Why Johnny Can't Add: the Failure of the New Math*¹⁶, to attack the way math teaching was done at United States.

This movement lead math teaching to degradation, to something like "back to basics" teaching where the primary aim was to return to teach the basic operations and the routines taught before the arriving of the New Math. Around the 1980's another reaction came from *National Council of Teachers of Mathematics*, the association of teachers of mathematics in the USA. This reform is now in course¹⁷, but not without protests, mainly from some circles of university professors.

7 – About the mathematics education curricula.

¹⁴ McCOLM, G. *A Metaphor for Mathematics Education*. Notices of the American Mathematical Society. Vol. 54. No. 4. April 2007.

¹⁵ BOREL, A. *Twenty Five Years with Nicolas Bourbaki, 1943-1973*. Notices of the American Mathematical Society. Vol. 45, No. 3. Pág. 374. March 1998.

¹⁶ KLINE, M. *Why Johnny Can't Add: The Failure of the New Math*. Random House Inc. New York. 1974.

KLINE, M. *Why the professor can't teach: Mathematics and the dilemma of university education*. St. Martin's Press. New York. 1977.

¹⁷ NCTM. *Principles and Standards for School Mathematics*. Ver: <http://standards.nctm.org/document/chapter2/index.htm>

We mentioned that there exist a large gap between the basic mathematical teaching and the mathematics at the research level. This is quite visible nowadays when we note that most of mathematics instructors at college level don't know about the mathematics supporting recent mathematical results as the classification of some Lie algebras (on the March 2007 news) or the proof of Poincaré Conjecture, proved by Grigori Perelman, who was awarded the Fields Medal at International Congress of Mathematicians, Madrid 2006. The importance of the Poincaré conjecture can be better appreciated when one discovers that six Fields Medals have been given to mathematicians working in subjects relating with this conjecture. Names as famous as John Milnor, Steve Smale, Charles Fefferman and William Thurston appear in this list¹⁸. No many colleagues know about this conjecture and about the implications that its proof has for science.

From 1936 up to now with the interruption around II World War, the Fields Medals have been granted to young mathematicians in specialties unknown to the general public such as: mathematical analysis, differential geometry, algebraic geometry, partial differential equations, topology and mathematical logic, among the most known. These areas are almost unknown to high school teachers and most of college professors also are not well informed about the developments of these mathematical fields. This appreciation is of course relating to my country, but for some information I have, the rest of Latin American countries are not so far from the mathematical level, where the Colombian teachers are.

Why, this kind of mathematics we are teaching is so far from the border of math research? One reason is that the mathematical contents in our curricula is quite old fashioned. We would need a radical transformation of the mathematical contents in traditional courses as arithmetic, algebra, geometry and infinitesimal calculus. Another reason could be that mathematics education is not changing at the same rate as research mathematics does. This means math education must be renewed permanently according to the needs of research in science and mathematics.

A congress as ICME (*International Congress of Mathematical Education*) is a natural forum where all math educators around the world could determine a real diagnostic of our situation relating to mathematical education and suggest what political and academic measures has to point out, so education at all levels can be improved.

7 – Some suggestions to take account.

I am not a mathematical education specialist. I am really seeing problems and possible solutions in my perspective of a mathematics professor with many years of experience at the college level. It is clear that I am not using the language and the specialized argot of professional educators; instead I am trying to suggest some naïve and common sense approaches to the formulation and possible solutions to problems relating separation between the mathematics we teach and the front of mathematics. These suggestions to improve mathematics education are closer related to

¹⁸ See, for instance the lecture notes by John Morgan at ICM, Madrid 2006, at :
SANZ-SOLÉ, M. et al, Editors. ICM, Madrid. 2006. Vol. I. *Plenary Lectures and Ceremonials*. European Mathematical Society.

developing countries as Colombia, and I think the way for a improving of mathematics education, is looking up for a better college and university education. The following are some suggestions.

- a) Reengineering in Teaching Teachers Schools.**
- b) More universities with Ph. D. programs and advanced institutes to make advanced mathematics research.**
- c) To improve mathematical level at universities, bringing talented and advanced professors from abroad.**
- d) Creation of Mathematical Institutes where university professors be educated.**
- e) To perform some kind of “compression” to introduce more mathematics in the basic mathematical courses.**
- f) Teaching mathematical analysis to prospective teachers instead infinitesimal calculus.**
- g) To introduce non Euclidean geometries at high schools and within the curriculum of prospective teachers.**
- h) Teaching modern mathematical logic to prospective teachers.**
- i) Teaching mathematical epistemology to prospective teachers of mathematics.**
- j) Bilingual education.**

The above notes are a compilation of several lectures delivered in 2007, mainly at XVI Congreso Nacional de Matemáticas, Medellín and at the VI Simposio Nororiental de Matemáticas, Bucaramanga, Colombia.

Armenia, January 18, 2008.