Mathematical Knowledge for Teaching (K-8):
Empirical, Theoretical, and Practical Foundations

Mark Hoover Thames, Laurie Sleep, Hyman Bass, and Deborah Loewenberg Ball
University of Michigan

Paper to be presented at ICME 11, TSG 27
Montérrey, Mexico
July 6 – 13, 2008

Author Note

The research reported in this paper was supported by grants from the National Science Foundation (Grant #s REC # 0126237, REC-0207649, EHR-0233456, and EHR-0335411) and the Spencer Foundation (MG #199800202). The authors thank members of the Mathematics Teaching and Learning to Teach Project and of the Learning Mathematics for Teaching Project for their help in developing aspects of this paper. Errors remain the property of the authors.

Mathematical Knowledge for Teaching (K-8): Empirical, Theoretical, and Practical Foundations

Mark Hoover Thames, Laurie Sleep, Hyman Bass, and Deborah Loewenberg Ball
University of Michigan
SEB 2400, 610 E. University, Ann Arbor, MI 48109-1259
Fax: 734-615-7441; Cell: 734-476-1990; mthames@umich.edu

Abstract

This paper describes a practice-based theory of "mathematical knowledge for teaching" (MKT) and reports on its empirical, theoretical, and practical foundations. Analysis of classroom instruction lays the foundation for uncovering the insufficiently understood mathematical requirements of teaching. This empirical work informs theoretical formulations of MKT and its practical use in the mathematical education of teachers. The paper concludes with a discussion of the benefits of a practice-based theory of mathematical knowledge for teaching and directions for future work.

Recent decades have witnessed numerous attempts to improve students' mathematics achievement. New standards, assessments, and curricula, however, do not automatically improve student learning; these resources must be put to use by teachers in classrooms (Cohen, Raudenbush, & Ball, 2003). Unfortunately, teachers generally have little opportunity to develop the mathematical knowledge and skill required to effectively implement new standards and curricula. Instead mathematical knowledge for teaching has typically been left to develop independently—and unreliably—from experience.

Common sense suggests that teachers' knowledge of mathematics impacts student learning, but studies attempting to link student achievement to their teachers' content knowledge have historically yielded small or mixed effects. (See Begle, 1979; Wayne and Youngs, 2003; Wilson, Floden, & Ferrini-Mundy, 2001.) The consistent lesson from this research has been that mathematical knowledge that is more closely related to practice—for instance, to specific curricula or to the work teachers do—is more likely to have a positive effect on teaching and learning. These findings have been echoed in interview and observational studies. (See Ball, Lubienski & Mewborn, 2001; Fennema & Franke, 1992.) These studies emphasize that teachers' knowledge of content, as suggested by Shulman and colleagues' (1986, 1987) introduction of "pedagogical content knowledge," must be closely tied to practice (Ball & Bass, 2003; Ball, Thames, & Phelps, 2007; Ma, 1999). Thus, it is not any type of mathematical knowledge that matters for teaching, but rather knowledge of mathematics that is usable for the work.

Our University of Michigan research groups also seek to answer questions regarding the composition and structure of mathematical knowledge for teaching (MKT), along three major lines of work: (1) by analyzing classroom teaching to see what mathematical knowledge arises in the work teachers do; (2) by developing theoretical domains and instrumentation for detecting, measuring, and categorizing mathematical knowledge for teaching; and (3) by teaching MKT to teachers and other professionals who work with teachers in various settings. This paper shows how each line of work contributes to the development and understanding of our practice-based theory of MKT.

Analyzing Classroom Teaching

To determine what teachers must know, previous research examined the elementary curriculum. We instead begin by investigating *practice* —the actual work of teaching—to uncover its mathematical demands (Ball & Bass 2003). This "job analysis" has revealed that elementary teaching is highly mathematical work; even seemingly general pedagogical tasks, such as listening to students or asking questions, require substantial mathematical knowledge and reasoning. In addition, our analyses have highlighted the centrality of mathematical practices in teaching—for example, representing ideas and procedures, interpreting and introducing mathematical notation, and providing and appraising explanations (Ball & Bass, 2000, 2003). Thus, in addition to knowing the curricular topics, teachers need be able to articulate important mathematical practices and cultivate their use by students.

To demonstrate the specialized ways that teaching depends on mathematical knowledge and skill, consider an example from division of fractions.[1] To teach this content, a teacher must, of course, be able to calculate the answer to problems such as:

$$\frac{5}{6} \div \frac{1}{3}.$$

Problems like this are in the student curriculum so must be part of the mathematical knowledge that elementary teachers need. Yet being able oneself to divide fractions is far from enough. Teachers need to know more than an effective algorithm, they must also be able to make the content accessible to students, interpret students' questions and productions, generate contexts

[1] A more elaborated version of this division of fractions example appears in Hill, Sleep, Lewis, and Ball (2007).

in which the content arises, and explain or represent the content in multiple ways—all tasks which require mathematical knowledge and reasoning. For example, a student might ask a teacher why the standard "invert-and-multiply" algorithm works, or why it is the second fraction that is "flipped over" rather than the first. Or, a student might wonder why the answer, 2½, is larger than both 5/6 and 1/3, because he thinks dividing should make the answer smaller. To respond to these questions, teachers must understand and be able to explain why the algorithm works, and be able to explain why and in what cases the quotient is larger than both the dividend and the divisor.

Instead of questioning why the standard procedure works, a student might claim that she does not need to learn how to invert and multiply because she has found an easier way to divide fractions—namely, to simply divide the numerators and divide the denominators:

$$\frac{5}{6} \div \frac{1}{3} = \frac{5 \div 1}{6 \div 3} = \frac{5}{2} = 2\frac{1}{2}.$$

This situation is not uncommon; students often develop their own methods of computation—some valid and some not—and teachers must be able to size up these alternative approaches. In this case, the teacher may need to determine whether the student's method for dividing fractions is mathematically valid or whether the correct answer was a lucky accident. And if the method is valid for this particular problem, will it work in general to divide any two fractions? Generating and testing examples and counterexamples entails mathematical knowledge and reasoning beyond the mathematics in the elementary curriculum. In fact, knowing to ask questions about a method's generalizability and efficiency is itself an example of the type of mathematical sensibility used in teaching. Teaching entails similar analyses when students arrive at incorrect answers. Teachers must be able to figure out what steps a student took to produce the error and the likely reasons it was made, and then, in light of this mathematical analysis, determine an appropriate response.

Categorizing and Measuring MKT

Our analyses of the mathematical demands of teaching inform our second line of work: developing theoretical domains and large-scale measures of MKT. We generated an initial set of domains from the literature and from our empirical study of teaching and learning. We then developed instruments to measure knowledge in these domains and conducted psychometric analysis to revise and refine the domains. The instruments contain multiple-choice questions reflecting common mathematics problems that teachers encounter in elementary school classrooms—for instance, evaluating students' mathematical claims, examining unusual solution methods, and determining how to best represent ideas or generate examples. These measures have been validated using cognitive interviews, studies of teaching practice, and links to student achievement. Our measures-development work has provided evidence that there is mathematical knowledge specific to teaching and that it can be measured (Hill, Schilling & Ball, 2004). Furthermore, it has shown that a teacher's mathematical knowledge for teaching is linked to the mathematical quality of instruction (Blunk, 2007) and is a significant predictor of gains in student achievement (Hill, Rowan, & Ball, 2005).

Based on this work, we have developed a framework for categorizing mathematical knowledge for teaching. We define MKT as the *mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to students*. To avoid a strictly reductionist and utilitarian perspective, however, we seek a generous conception of "need" that allows for the perspective, habits of mind, and sensibilities that matter for effective teaching of the discipline. The framework currently distinguishes four main domains of MKT: (1) *common content knowledge*; (2) *specialized content knowledge*; (3) *knowledge of content and students*; and (4) *knowledge of content and teaching* (Ball et al., 2007).

We define *common content knowledge (CCK)* to be the mathematical knowledge and skill used in teaching yet also used in settings other than teaching. For example, teachers need to be able to *do the work* that they assign their students. This is mathematical knowledge and skill that others, besides teachers, have as well—thus, it is not special to the work of teaching.

In contrast, *specialized content knowledge (SCK)* is the mathematical knowledge and skill uniquely needed by teachers in the conduct of their work. For instance, in looking for patterns in student errors or in sizing up whether a non-standard approach would work in general, teachers have to do a kind of mathematical work that others do not. Many of the everyday tasks of teaching are distinctive to this special work, such as choosing an example to make a specific mathematical point, modifying tasks to be either easier or harder, or explaining why we find common denominators when adding fractions (but not when multiplying them). Teaching also involves making compressed features of particular content, for example, place value, visible to and learnable by students.

The third domain, *knowledge of content and students (KCS)*, combines knowing about students and knowing about mathematics. Teaching depends on anticipating what students are likely to think and what they will find confusing. Teachers need to predict what students will find interesting and motivating, and to anticipate what students are likely to do with a task, whether they will find it easy or hard. They must also be able to hear and interpret students' thinking as expressed in the ways that pupils use language. Each of these tasks requires an interaction between mathematical understanding and familiarity with students' mathematical thinking.

The last domain, *knowledge of content and teaching (KCT)*, combines knowing about teaching and about mathematics. Many of the mathematical tasks of teaching require a mathematical knowledge of the design of instruction. Teachers need to sequence particular content for instruction, choosing initial examples and then others to take students deeper into the content. They need to make instructional decisions about which student contributions to pursue and which to ignore or save for a later time. They also need to evaluate affordances of different models for place value, what each can be used to make visible, as well as how to deploy them effectively. How is money different from coffee stirrers bundled with rubber bands, or base ten blocks from "unifix" cubes? Each of these *can* correctly represent operations with multi-digit numbers, but with significant pedagogical differences. Knowing how those differences matter for the development of the topic is part of what we call knowledge of content and teaching.

Our definitions and examples make clear that MKT elaborates pedagogical content knowledge, rather than replaces it. For instance, the last two domains—knowledge of content and students and knowledge of content and teaching—coincide with the two central dimensions of pedagogical content knowledge identified by Shulman (1986). However, we also see our work as developing in more detail the fundamentals of *subject matter knowledge for teaching* by elaborating sub-domains and by measuring and validating knowledge of those domains.

Specifying and Teaching an MKT Curriculum

In our third line of work, we have developed and evaluated approaches to helping teachers and other professionals learn MKT. For example, we teach mathematics content and methods courses for teacher education students, facilitate MKT-focused professional development for practicing teachers, organize regional study groups with fellow mathematics teacher educators, and offer institutes for teacher educators, mathematicians, and other teacher developers. In doing this work, we have developed a wide range of tasks and materials that create opportunities for learning MKT. Doing so has placed practical demands on our theoretical ideas and also illuminated their expansion.

We have taken a problem-based approach to teaching MKT, finding and designing practice-based tasks as described by Ball & Cohen (1999) and Smith (2001). Just as our empirical work starts with practice to study its knowledge demands, our instructional tasks situate opportunities to learn MKT in the contexts of its use, for example using records of classroom practice—including video, student work, and curriculum materials—to provide opportunities to engage in the kinds of mathematical thinking, reasoning, and communicating used in teaching.

Practice-based approaches, while increasingly common in methods courses and professional development settings, are not as widely used in *mathematics* courses for teachers.[2] Although some mathematics courses and textbooks for teachers do relate mathematical knowledge to aspects of practice, the typical approach is to first teach mathematics and then apply that knowledge to teaching, rather than to teach mathematics that is directly situated in and taught through engagement in the work of teaching. That is, in most courses, the mathematics is in the foreground, with teaching in the background as a context to justify or apply the content being taught. This latter perspective, if exclusive, risks preparing teachers who are very good at doing mathematics in their courses, but who cannot use their mathematics knowledge in actual teaching situations. We argue that the mathematics taught in courses for teachers should be the mathematics required for the work of teaching, and that this mathematical knowledge for teaching should be integrated with and learned in the contexts of its use in practice. Thus, we place the work of teaching is in the foreground, and mathematics is learned through its manifestations in practice. This is not unlike the way that mathematics is learned for other mathematically intensive professions.

The matrix below represents our current approach to specifying a curriculum for MKT, where rows represent mathematical topics found in the school curriculum and columns represent mathematical tasks of teaching. For example, an instructional activity that asks teachers to model the standard subtraction algorithm with base ten blocks is located in the place value and operations row, and the main tasks of teaching addressed are representing and recording mathematical ideas, explaining mathematical ideas, and using mathematical language. As teachers learn to engage in these tasks of teaching, they simultaneously learn why the subtraction algorithm works, gain a

[2] On the flip side, practice-based methods courses and professional development often fail to maintain adequate attention to the mathematics at hand. It is the tension between compelling non-mathematical issues of practice and compelling practice-irrelevant mathematical issues that we seek to resolve.

deeper understanding of place value and of meanings of subtraction, and develop skills of representing, explaining, and using mathematical language in and for instruction.

Mathematical Tasks of Teaching

Mathematical Topics	Explaining mathematical ideas	Representing & recording mathematical ideas	Interpreting & eliciting mathematical thinking	Using mathematical language	Generating & analyzing alternative solutions methods	Analyzing errors	Creating, analyzing, & modifying mathematics tasks
Place value & operations							
Fractions							
…							

Our work to teach MKT both uses and contributes to our theoretical and empirical work. For example, one question we have become interested in as we design MKT activities is: What makes a "good" MKT task? In particular, what is the difference between a good mathematics task and a good MKT task? To pursue this question, we have begun to identify features, such as the following, that make a task well suited for teaching MKT:

- Unpacks, makes explicit, and develops a flexible understanding of mathematical ideas that are central to understanding the school curriculum.
- Provokes a stumble due to a superficial "understanding" of an idea
- Lends itself to alternative/multiple representations and solution methods
- Provides opportunities to engage in mathematical practices central to teaching (e.g., explaining, representing, using mathematical language, analyzing equivalences, proving, proof analysis)
- Provides opportunities to engage in teaching practices that are central to mathematics teaching (e.g., interpreting others' thinking, posing questions, writing math on the board, talking)

This list is by no means exhaustive, and we are not yet sure if a particular MKT task would need to have all of these features. We share this here not as a conclusive list of features, but as an example of how our practical work informs and influences our empirical and theoretical work.

Conclusion

Our evolving practice-based construct of MKT has been positively related to mathematical quality of instruction and to student learning. It includes sub-domains that identify mathematical knowledge that is special to the work of teaching, and can inform the design of curriculum for teacher education and professional development. This is work in progress. Although we have provisionally identified sub-domains of MKT, further work is needed to specify what constitutes and distinguishes them. We also need to uncover the ways in which teaching entails mathematical reasoning and action rather than inert "knowledge," and how MKT relates to other important capacities in teaching. We are also investigating the extent to which our construct of MKT is culturally specific (Cole, in progress; Delaney, 2008), or dependent on teaching styles.

References

Ball, D. L., & Bass, H. (2000). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In J. Boaler (Ed.), *Multiple perspectives on the teaching and learning of mathematics* (pp. 83-104). Westport, CT: Ablex.

Ball, D. L., & Bass, H. (2003). Toward a practice-based theory of mathematical knowledge for teaching. In B. Davis & E. Simmt (Eds.), *Proceedings of the 2002 annual meeting of the Canadian Mathematics Education Study Group.* Edmonton, AB: CMESG/GCEDM.

Ball, D. L., & Cohen, D. (1999). Developing practice, developing practitioners: Toward a practice-based theory of professional education. In G. Sykes & Darling-Hammond (Eds.), *Handbook of policy and practice* (pp. 3-32). San Francisco: Jossey Bass.

Ball, D. L., Lubienski, S., & Mewborn, D. S. (2001). Research on teaching mathematics: The unsolved problem of teachers' mathematical knowledge. In V. Richardson (Ed.), *Handbook of research on teaching* (4th ed.) (pp. 433-456). New York, NY: Macmillan.

Ball, D. L., Thames, M., & Phelps, G. (2007). *Content knowledge for teaching: What makes it special?* Unpublished manuscript.

Begle, E. G. (1979). *Critical variables in mathematics education: Findings from a survey of the empirical literature.* Washington, DC: Mathematical Association of America and National Council of Teachers of Mathematics.

Blunk, M. L. (2007). The QMI: Results from validation and scale-building. Paper presented at the annual conference of the American Educational Research Association.

Cohen, D., Raudenbush, S. W., & Ball, D. L. (2003). Resources, instruction, and research. *Educational Evaluation and Policy Analysis, 25*(2), 119-142.

Cole, Y. (in preparation). *Mathematical knowledge for teaching in Ghana*. Doctoral research in progress.

Delaney, S. (2008). *Adapting and using U.S. measures to study Irish teachers' mathematical knowledge for teaching*. Unpublished doctoral dissertation, University of Michigan, Ann Arbor, Michigan, U.S.A.

Fennema, E., & Franke, M. L. (1992). Teachers' knowledge and its impact. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 147-164). New York: MacMillan Publishing Co.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*(2), 371-406.

Hill, H. C., Schilling, S. G., & Ball, D. L. (2004). Developing measures of teachers' mathematical knowledge for teaching. *Elementary School Journal, 105*(1), 11-48.

Hill, H. C., Sleep, L., Lewis, J. M., & Ball, D. L. (2007). Assessing teachers' mathematical knowledge: What knowledge matters and what evidence counts? In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 111-155). Charlotte, NC: Information Age Publishing.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review, 57*, 1-22.

Smith, M. S. (2001). Practice-based professional development for teachers of mathematics. Reston, VA: National Council of Teachers of Mathematics.

Wayne, A. J. & Youngs, P. (2003). Teacher characteristics and student achievement gains: A review. *Review of Educational Research, 73,* 89-122.

Wilson, Floden & Ferrini-Mundy. (2001). *Teacher preparation research: Current knowledge, gaps, and recommendations.* Center for the Study of Teaching and Policy. Executive Summary.