COGNITIVE AND METACOGNITIVE PROCESSES OF PRE-SERVICE MATHEMATICS TEACHERS WHILE SOLVING MATHEMATICAL PROBLEMS

By
Omar Hernández Rodríguez, Ed.D.
Wanda Villafañe Cepeda, Ed.D.
University of Puerto Rico, Río Piedras Campus

A phenomenological study about mathematical problem solving is described. Eight pre-service mathematics teachers participated; six were studying to become teachers at elementary school -4th to 6th grades- and two at high school -7th to 12th grades-. The data was obtained through long interviews, thinking out loud problem solving sessions and retrospective interviews that took place immediately after the problem solving sessions. The objective of the long interview was to determine the participants' beliefs and declarative knowledge about this topic. The objective of the problem solving sessions was to determine the type of representation, strategies, and control processes that the participants use when solving problems. During the retrospective interview, the participants had the opportunity to reflect about their performance. These techniques allowed the investigators to obtain a comprehensive description of the phenomenon.

INTRODUCTION

Problem solving is recognized by some experts as a fundamental process for the mathematical development of students. Some of the most influential organizations in mathematics education have recognized the importance of problem solving in school mathematics (National Council of Teachers of Mathematics, 2000; American Association for the Advancement of Science, 1993). The Mathematics Program of the Department of Education of Puerto Rico, states in its mission that solving problems is important as an aim and a mean for the learning of mathematics. (DEPR, 2003).

Regardless of the recommendations and the importance recognized by experts and educational organizations, what happens in the mathematics classrooms is completely different. After finishing their education, some teachers ignore what they learned, and their practice reflects more their experience as students rather than what they were taught in their pedagogy classes (Skott, 2001, as cited by Liljedahl, et al. 2007). Some authors ascribe this situation to the teachers' knowledge about the discipline (Ball, 1990; Leonard and Joergensen, 2002; Van Dooren, et al., 2003); others, to teachers' affective and metacognitive factors, including their beliefs (Grows y Good, 2002; Liljedahl, et al. 2007; Mewborn y Cross, 2007).

Education schools prepare their students in three areas: discipline knowledge, knowledge of the fundamentals of education, and the teaching of the discipline. However, the time dedicated to the study and development of metacognitive strategies is minimal. We hope that this research can provide the insight required to make curricular changes, so that the study and development of the metacognitive processes can be included in the curriculum of prospective teachers.

REVIEW OF LITERATURE

Problem solving has been the object of many researches; however, there are still unanswered questions about it (Lester, 1994). The study of problem solving started with the publication of *How to Solve It* (Polya, 1945), which was followed by a series of research studies about the effectiveness of the use of general problem-solving strategies to learn mathematics. Later, research was based on the information processing theory with the objective of determining how the experts solve problems (Schoenfeld, 1985; Schoenfeld & Herrmann, 1982; Silver & Marshall, 1990). Recently, with the advent of the constructive learning theory, there has been a new wave of research that focuses in other aspects, such as the role of the metacognition, the students and teachers' beliefs, and affective and social influences when solving a problem (Garafalo and Lester, 1982; Hernández Rodríguez, 2002; Maqsud, 1997; Santos Trigo, 1995;

Schoenfeld, 1987, 1989, 1992; Swanson, 1990, 1992). It has been established that, even if the students possess the mathematical knowledge, it is very difficult for them to use that knowledge in new situations (Santos Trigo, 1995; Schoenfeld, 1985). Moreover, in few occasions they are able to use specific strategies to solve mathematical problems (Hernández Rodríguez, 2002). It also has been concluded that, in addition to the specific knowledge, students need other strategies to solve problems (Polya, 1945; Santos Trigo, 1995; Schoenfeld, 1985). More specifically, it has been established that students can benefit more if they learn general problem-solving strategies (Hembree, 1992; Lawson, 1990; Silver and Marshall, 1990).

Flavell (1976) defined metacognition as the knowledge that people have about their own cognition and the self-regulation processes of the cognitive processes. Later, this definition was expanded to include the students' beliefs about themselves, about mathematics, about the task, and about the strategies required by a given situation (De Corte, Greer and Verschaffel, 1996; Garafalo and Lester, 1985; Greeno, Collins and Resnick, 1996; Lampert, 1990; Schoenfeld, 1987). Stemberg (1998) considered that metacognition is part of the human abilities and that it is indispensable in the formation of the discipline.

Lamper (1990) found that students believe knowing mathematics is about remembering and applying certain rules correctly in a problem, and that the only correct answer is the one the teacher gives. These beliefs have a generally negative effect in the way the students perform when solving mathematical problems. Schoenfeld (1987) stated that students' mathematical beliefs are important to help or interfere in the process of problem solving. For example, he found that students thought that a mathematical problem has to be solved in less than ten minutes and this belief lead them to abandon it if they do not get a solution quickly.

Problem solving in pre-service mathematics teachers had been studied in different aspects. Bjuland (2004) did a study with 105 pre-service teachers in which they reflected on their own learning process when solving geometric problems in a collaborative way. Chapman (2005) conducted a qualitative study intended to determine the knowledge that pre-service teachers have about problem solving and the effects of the incorporation of reflection and inquiry processes to the improvement of this knowledge. Cadenas (2007) did a study intended to determine the lacks, difficulties and errors that pre-service teachers have in their mathematical knowledge previous the start of university studies.

Mathematics teachers' beliefs and their relationships with the student learning process have been studied by Mewborn and Cross (2007). Teachers' cognitive and metacognitive preferences have been studied by Leikin (2003), and Grouws and Good (2002). In regard to pre-service teachers, Liljedahl, Rolka and Rösken (2007) studied the affective aspects of problem solving, and Van Dooren, Verschaffel, and Onghena (2003) investigated the evolution of pre-service mathematics teacher's cognitive preferences.

Lampert (1990) said that the school has a major responsibility in the development of student's beliefs about the meaning of knowing mathematics and how to do mathematics. These beliefs are created after so much time of looking, listening and practicing mathematics in schools. Since beliefs are mental constructions originated by previous experiences and social interactions, we can argue that student's beliefs are mostly influenced by their teacher's beliefs. Part of the difficulty that students show with problem solving can be explained by their teacher's beliefs about it (Goss, 2006; Mewborn and Cross, 2007). At the same time, the mathematics teachers' beliefs are the result of their school experiences and the knowledge acquired as education students. The dialectic interaction between beliefs and professional evidence are the object of this investigation.

With regard to the representations that are used by pre-service teachers when solving problems, it has been found that the numeric-table representation dominates over the algebraic and geometric ones (Presmeg and Nenduradu, 2005). Moreover, Mousolide and Gagatsis (2004) found that teachers do have difficulties constructing geometric representations.

It is important to remark the importance of mathematic content in the problem solving process. Some authors claim that a good knowledge of the discipline is a great factor when dealing with problem solving (Ball, 1990; Cadenas, 2007; Leonard and Joergensen, 2002; Van Dooren, et al., 2003). Ma (1999) concluded that teachers that have a deep understanding about mathematical concepts can create a problematic situation related to the mathematical concept more easily, which implies that a deeper understanding of the mathematical concepts redounds in a better pedagogical knowledge.

To think about the beliefs and the way that pre-service teachers solve problems will allow the researchers propose educational environments that promote the construction of beliefs and knowledge that favor problem solving.

RESEARCH QUESTIONS

This investigation was guided by the following questions:

- What beliefs do pre-service mathematics teachers have about mathematical problem solving?
- Which kind of external representations (iconic or symbolic) do pre-service mathematics teachers use when solving a non typical problem?
- What kind of strategies (general or specific) do pre-service mathematics teachers use when solving a non typical problem and in which circumstances do they use them?
- How does self-regulation intervene during the different stages of problem solving?

DEFINITION OF TERMS

A non typical mathematical problem is a situation that has to be modeled to find an answer to a question that derives from the same situation and which solution is not straightforward. (Parra, 1991).

Problem solving refers to the coordination of knowledge, previous experiences, and intuition in an effort to find a solution that is unknown (Parra, 1991). Operationally, it is the set of all written and verbal processes used by the student to find the answer to a problem.

The cognitive processes that will be studied in this investigation are the construction of the problem's representation and the strategy selected and used to solve a problem.

An external representation is a stimulus to the sense, generally in the form of drawings, diagrams, graphics, models or other formal symbolic systems (Janvier, Girandon and Morand, 1993).

A general strategy is a technique that can be applied to various knowledge domains and that serves as a guide to solve a problem. Some general strategies are trial and error, finding a pattern, constructing a table, using analogies, using auxiliary elements and backward working. A specific strategy is a technique that can be used to solve a problem in a specific domain.

Metacognitive processes include the beliefs and the processes of self-regulation and control. A belief is a made-up explanation that a person has about a specific field of knowledge and that determines the way the person conceptualizes and fulfills on it (Schoenfled, 1992). A belief can be about itself (De Corte, Greer and Verschaffel, 1996), the area of study - mathematics, in this case- (Greeno, Collins and Resnick, 1996) or the task to be done (Garofalo and Lester, 1985).

The self-regulation or control is an ordered process used by a person to control its own cognitive activity, and, in this way, ensure the accomplishment of the cognitive objective (Schraw and Graham, 1997). A person that can control its cognitive activity can make predictions, elaborate a plan before starting to solve a problem, pay attention to all the components of a problem, question the process, value the products and the efficiency of the execution, and review, change, and abandon unproductive strategies or plans (Garofalo and Lester, 1985; Schraw and Graham, 1997).

METHODOLOGY

In the present study, a constructivism point of view is assumed. This considers that human beings build their own knowledge, and that cognitive and metacognitive processes take part inand that cognitive and metacognitive processes take part in the construction of such knowledge (Flavell, 1976; Noddings, 1990; von Glasersfeld, 1990). In addition, it is also recognized that some social and emotional factors intervene in the construction of knowledge (Greeno, Collins, and Resnick, 1996). To have access to the cognitive and metacognitive processes, different methodologies were used to describe what was happening in the minds of the participants. The research design used responds to the necessity of gaining access to the field of the perceptions to explore what is happening when non typical problems are solved from the participant's perspective.

This is a phenomenological study about cognitive and metacognitive processes that pre-service mathematics teachers exhibit about solving non typical mathematical problems. It was intended to find deep meanings, understandings and attributes of the phenomenological target that was studied. The meaning that various people ascribe to the concept or phenomenon is described (Creswell, 1998; Moustakas, 1994), the experiences that people have had with the phenomenon are explored, and the essential structure or invariant in which underlies the meaning of the experience is outlined. In this way, the intentionality of the conscience is described where the experiences contain the external appearance as well as the internal conscience (Moustakas, 1994). From this point of view, the phenomenon gains meaning through people experiences with it. To have access to the essence, product of the range of interactions that the people have had with the phenomenon, we analyzed the memory, the image and the significance that they attribute to it (Creswell, 1998; Morse, 1994).

Participants

The participants were university students enrolled in the teachers formation program of a public university in Puerto Rico, specifically, those who majored in mathematics education at elementary level or secondary level. Participation was on a voluntary basis. A public announcing was done to the university community, requesting the participation of volunteers; eight people applied, all of them females. The candidates assisted to an orientation, in which the nature of their participation was discussed, as well as the observance, by the investigators, of the university rules concerning the protection of human rights when students participate in an investigation.

Data collection

The procedures used to collect the information were descriptive and qualitative, designed to describe a wide range of internal and external activities. The techniques used were: long interview, thinking-out-loud problem solving session, and a retrospective interview right after the problem solving session. These techniques let the participants reflect about the theme, which will contribute to their formation as teachers.

The long interview allowed the access to the meanings that participants had about problem solving and it was possible to describe the beliefs that they have about this process. After that, the participants solved four non typical mathematical problems. The participants were instructed to solve the problems thinking out loud. This technique allowed the access to the cognitive and metacognitive processes the participants went through. Immediately after each problem solving session, a retrospective interview was made in which the participant had the opportunity to reflect on their execution in the problem solving process. Thus, not only was explored what was happening in the participant's minds, but also the problem solving execution and the participant's thoughts about it. In this way, there were three sources of information that allowed the triangulation of the data, allowing the researches to conclude which kind of representation, strategy, metacognitive process of control and beliefs the participants have about solving non typical mathematical problems.

Problems

The problems used have the characteristic that they are sufficiently challenging to produce metacognitive behavior, but at the same time could be solved by the students with the mathematical knowledge they had from their mathematical classes (Goos and Galgraith, 1996). In addition, the problems can be represented in different ways and solved using diverse strategies. The problems that were used for this analysis were problem 2 and 4.

<u>PROBLEM 2</u>

A square and a rectangle have the same area. The square diagonal has a longitude of $8\sqrt{2}$ inches. If the width of the rectangle is 4 inches, what is the length of the rectangle?

<u>PROBLEM 4</u>

A candy sale is organized with the purpose of raising funds for the Children Cancer Association. Olga, who is engaged in the cause, wants to sell 27 chocolate bags. There are two different kinds of chocolate: with almond and with strawberry. Each bag of chocolate with almond has 8 bars and each bag of chocolate with strawberry has 9 bars. If Olga has a total of 232 chocolate bars, how many bags of each kind of chocolate does Olga have?

Analysis

The analysis of the collected data was enriched by the comprehension reached by the investigators, and complemented by the review of literature and their experience as professors and investigators. All the interviews were made and transcribed by the investigators. The fidelity of each transcription was corroborated with the audio records and simultaneous reading of what was transcribed.

Particularly, the analysis that the investigators did of the long interviews let them find an answer to the students’ beliefs question. The retrospective interview analysis let them determine which self-regulation processes were used when the participants solved each problem and helped them complement and contrast all the information obtained in the long interviews and problem solving sessions.

On the other hand, the analysis of the problem solving session provides plenty information about how the students construct the representation of each problem, the strategies that they use and the self-regulation strategies they showed in the solution process. The investigators used the audio record of each problem solving session, all the computations the participants made during the process and the transcriptions of their observations as data to analyze.

RESULTS AND DISCUSSION

The results obtained are organized following the research questions. The extensive interviews were used to establish the students’ beliefs about mathematical problem solving, and the analysis of the solutions of problems 2 and 4 to establish the representations, the strategies and the processes of self-regulation used by the participants. The retrospective interviews were used to triangulate the information.

Beliefs

In general terms, the participants consider themselves very good mathematical problem solvers. This indicates great self-confidence although in many occasions they expressed having difficulties solving the problems. They attribute their good disposition mainly to affective reasons. The majority declared that problem solving represents a challenge and it motivates them; other reasons for their good disposition were that they were interested in mathematics or that they liked to solve problems.

The participants characterized a mathematical problem as an uncertain situation because they do not know the subject matter or the method needed to solve it. A problem requires deeper analysis; several pieces of knowledge take part and must be used simultaneously in the process of solving a problem. In contrast, they already know what to do when it comes to solving an exercise, since exercises are solved with some well-known algorithm.

Just as in the investigation of Chapman (2005), the majority point out that the steps used to solve a problem are: reading the statement, identifying the given data, determining what is asked and solving it. The participants assigned greater importance to the understanding of the problem and less to the process of solving it. Still more, some participants assigned great importance to the reading of the problem, because they think that the way to solve the problem is ciphered in the statement.

The students declared preferring the arithmetical and algebraic strategies over the graphical strategies. All the participants indicated that they verified the problem to know if the answer were correct, however this was not observed when they solved the proposed problems.

Some indicated that, when they are solving a problem and do not know how to follow, they returned to read and "reread" the statement. Others indicated they abandoned it temporarily and returned to it later on. One student indicated that she reviewed the notes to see if she had solved a similar problem previously, another one mentioned that she tried to get help. Half of the students indicated that "they analyzed" the problem when they did not know how to follow ahead. For them, a problem is difficult when: they cannot solve it in the first attempt, or they do not understand it when they read it the first time. It is also difficult when it contains too much information or when different operations are needed in the solution process.

When asking how they considered that problem solving should be taught at school, the majority indicated that it must be integrated more frequently in class. They also point out that problem solving should not be a separate or isolated topic; instead it must be related to daily life events. Also they indicated that the teachers must give the students more problems to solve, that is, more frequent practice, thing that many of them did not get when they were students. This aspect agrees with the obtained by Grouws and Good (2002), which found that problem solving was not very frequent subject in the classes of mathematics of the observed teachers.

One of the participants narrated a negative experience that she had had with a university professor when she was solving a problem that involved roots. Nowadays it is very difficult for her to solve problems that includes these, even more, she feels uneasy whenever she faces a problem with that include roots. In fact, she could not solve problem 2, which included a squared root.

Representations

Immediately after reading problem number 2, the students used graphs to represent it.

They drew a square, a rectangle and some marked the measurement of the diagonal of the square. In addition, three students represented in iconic form the condition that the area of the square was equal to the area of the rectangle. Nevertheless, only two could establish the connection between the graphical representation and the algebraic one, which limited the use of this one. This last point converges with the indicated by Gagatsis, Elia, and Kyriakides, 2003; as cited by Mousoulides and Gagatsis (2004), in the sense that the pre-service teacher that participated in their study could not make the connection between the graphical and the algebraic representation.

Several difficulties in the representation of the problem appeared. These can be classified as difficulties that came from the invention of conditions from the data, the omission of data that the problem provided, and others whose origin is mathematical. First of all, three students used $8\sqrt{2}$ as the diagonal of the rectangle. Second, a student did not take into account the piece of information regarding the fact that the two figures had the same area; this prevented her from solving the problem. Finally, the main mathematical error in the representation was that the students thought that the measurement of the diagonal of the unit square is one. This made them construct a square whose sides measured the same as the diagonal.

The representation of problem four occurred in different form. Half of the students underscored the relevant information in the statement of the problem; the other half rewrote it in the worksheet. The majority used numerical representations, only 2 made algebraic representations. After reading the problem, the students began to conduct arithmetical operations with the numbers that the statement provided, which indicates that there is no understanding of the problem immediately after the reading.

The representation used is privileged by the situation of the problem. In problem number two the graph prevailed, whereas in problem number four the numerical one, nevertheless, the participants could not connect this first representation to the mathematical content that allowed them to solve the problem. Once the initial representation was constructed, the participants did not change it, which shows little flexibility to change plans, although in several occasions the students returned to reread the statement.

Strategies

To solve the second problem, three participants tried to construct the figures from the diagonal. None were successful because of the errors they had committed in the initial representation of the problem. Two participants used the Pythagorean Theorem, one succeeded and the other not, since she used it in a rectangle, in which she had labeled $8\sqrt{2}$ as the measure of the diagonal. Two students did not make any attempt solve the problem, nevertheless, one guessed an answer. The other student indicated the process that she would have used to solve the problem; nevertheless, she indicated that she did not know how to calculate the side of the square.

With respect to the fourth problem, seven of the participants used the strategy of trial and error. From these, three arrived at the correct answer. They used "educated estimates", that is, they tried some set of possible values and adjusted them according to the results they were obtaining, eventually arriving to the correct answer.

Five participants performed calculations with the numbers given in the problem. They carried out operations such as $232 \div 2 = 116$; $116 \div 8 = 14.5$; $116 \div 9= 12.88$ or $27 \div 2=13.5$, with the hope that the results would fit some of the given information. This agrees with the findings of Kieran, 1992; and Linchevski and Herscovics, 1996; as cited by Van Dooren, Verschaffel, and Onghena (2003), whom indicated that the students preferred to perform arithmetical operations with the known numbers, the meaning of such operations remaining invariably connected to what the students perceive to be the context of the original problem.

One participant used the specific strategy of system of equations; in particular, she set up a system of 2 linear equations with 2 variables, obtaining the correct answer. This participant was majoring in secondary education in mathematics. This finding agrees with the obtained one Van Dooren, et al. (2003), whom indicated that secondary school pre-service teachers prefer to use of algebra.

Several investigators have documented the difficulties that the students have when they face algebra for the first time and, in specific, the solution of algebraic problems (Filloy & Sutherland, 1996; Herscovics & Linchevski, 1994; Kieran, 1992; Lee & Wheeler, 1989; Sfard & Linchevski, 1994; as cited by Van Dooren, et. al, 2003). Some educators and investigators have suggested that a way to solve these difficulties is "to algebrazied" the elementary mathematics curriculum (Ainley, 1999; Davis, 1985; Discussion Document for the Twelfth ICMI Study, 2000; Kaput, 1995; Swafford & Langrall, 2000; Vergnaud, 1988; as cited by the authors). They argued that, early in the school mathematics education, the arithmetical activities can and must gradually be attended with an algebraic meaning with the purpose of emphasizing the inherent algebraic characteristics. Incorporating this recommendation to the curriculum of pre-service elementary teachers will possibly help them extend their repertory of strategies to solve problems successfully.

Self-regulation

During the problem solving sessions the participants showed little metacognitive activity, for example, they did not express their familiarity with the problem, nor stated its level of difficulty. In the case of the geometry problem, a student declared with disappointment “this is a geometry problem” when reading the problem. In the retrospective interview it was possible to verify that she had trouble with the subject. Another student stated that everything related to radicals caused her anxiety due to an unpleasant experience she had when she was learning them. The other students did not indicate their confidence (or lack of it) of solving the problem.

In few occasions the participants showed evaluation strategies of the representation or the strategy they were using. For example, very few participants changed the initial representation and even fewer changed the strategy that they selected initially even if it did not produce the awaited results. The evaluation of the progress towards the solution occurred more in problem four than in problem two, since the students used the values of the statement of the problem or some that they considered important to the solution. Another element that was used to verify the progress towards the solution of the problem was the appearance of decimal numbers in the results of the operations. In this aspect, it is important to stress that although the initial performed operations did not have sense, these helped them start making use of the trial and error strategy with initial values closer to the actual solution.

Having taken a geometry course did not help them solve problem number two. Similarly, the familiarity with problem number four did not help them arrive to the correct answer. This is explained since it is difficult for students to apply their mathematical knowledge to a novel situation (Hernandez Rodriguez, 2002; Santos Trigo, 1995; Schoenfeld, 1985; Selden, Selden, Mason, 1994).

It was observed that some of the participants who had been more time working in the problem tend to ignore the initial conditions. This can be explained in two ways: they are approaching the time limit that they are supposed to expend on it or for them it is indispensable to give an answer (Schoenfeld, 1989).

CONCLUSIONS

1. The participants characterized the mathematical problem as an uncertain situation because they do not know what it is about or the method in question to solve it. A problem requires deeper analysis; several pieces of knowledge take part and must be used simultaneously in the process of solving a problem. In contrast, they already know what to do when it comes to solving an exercise, since exercises are solved with some well-known algorithm.
2. The participants declared to prefer the arithmetical and algebraic strategies over the graphical strategies.
3. All the participants indicated that they verified the problem to know if the answer was correct, although this was not observed in the problem solving sessions.
4. Most of the participants indicated that problem solving must be integrated more frequently in mathematics classes. Problems solving should not be studied as a separate or isolated topic, and it must be related to daily life situations.
5. Participants assigned great importance to the reading of the problem because they think that the way to solve it is ciphered in the statement.
6. Participants used a graphical representation for problem number two and a numeric representation for problem number four.
7. There were several difficulties in the representation of the problems. These can be classified as difficulties that came from the invention of conditions from the data, the omission of data that the problem provided, and others whose origin is mathematical.
8. With regard to the geometry problem, three participants tried to construct the figures from the diagonal. This strategy was not successful because of the errors they committed in the initial representation of the problem.
9. Two participants used the Pythagorean Theorem to solve problem number two. One was successful and the other not, since she used it in the rectangle and she had labeled its diagonal as $8\sqrt{2}$.
10. With regard to problem number four, seven of the participants used the strategy of trial and error. From these, 3 arrived at the correct answer. The method used was "educated rough estimates", that is, they tried some set of possible values and adjusted them according to the results they were obtaining, eventually arriving to the correct answer.
11. The participants showed little metacognitive activity. They did not express their familiarity with the problem, nor the level of difficulty of the same. They did not indicate their confidence to solve the problem.
12. In few occasions, the participants showed evaluation strategies.
13. It is difficult for the participants to apply the mathematical knowledge to solve unfamiliar situations.
14. Participants possessed declarative knowledge about problem solving; however, it was difficult for them to use it to solve the problems posed.

Educational implications

Pre-service mathematics teachers must be exposed frequently to problem solving in their mathematics classes, so that they develop the necessary skills to solve and teach appropriately to their students. The use of diverse representations should be stimulated to fortify the connection between them, so that they can use the one that is suitable at the appropriate time.

Finally, it is essential to foment the use of algebraic strategies in the pre-service elementary mathematics teacher, so that the arithmetical activities can be attended with an algebraic meaning.

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