Abstract ICME 11

TWO CULTURES OF MATHEMATICS IN HISTORICAL AND EDUCATIONAL PERSPECTIVE

Profª. Drª. Gladys Denise Wielewski[1]
Universidade Federal de Mato Grosso-Cuiabá
gladysdw@brturbo.com.br

In a widely cited paper T. Gowers claims that there exist nowadays two different "cultures" in mathematics. Gowers writes: "Loosely speaking, I mean the distinction between mathematicians who regard their central aim as being to solve problems, and those who are more concerned with building and understanding theories".

Now these two orientations have existed throughout the history of mathematics. It is well known, for example, that Euclid's *Elements* contain construction problems and theorems. In Proclus commentary on Euclid one reads already:

"Now some of the ancients, however, such as the followers of Speusippus, insisted on calling all propositions 'theorems', considering 'theorems' to be more appropriate designations than 'problems' for the objects of the theoretical sciences since these sciences deal with eternal things. There is no coming to be among eternals, and hence a problem has no place here, proposing as it does to bring into being or to make something not previously existing - such as to construct an equilateral triangle ..... Others, on the contrary, such as the mathematicians of the school of Meneachmus, thought it correct to say that all inquiries are problems" (Proclus Commentary, Princeton UP 1970, 62, 77.7).

Proclus thinks that both parties are right: "The school of Speusippus is right because the problems of geometry are of a different sort from those of mechanics". On the other hand, "without entering into matter there is no discovery of theorems, that is, into intelligible matter" (*loc.cit.*).

Another example that comes readily to mind is the difference between Descartes, who was exclusively concerned with problems and methods, on the one side and Leibniz, who created the notion of formal mathematical proof in the modern sense being inspired by Cartesian analytical geometry, on the other side (HACKING, 1980). Leibniz believed truth to be established by formal theory and proof, Descartes

[1] Professora do Departamento de Matemática e da Pós-Graduação em Educação.

considered proof irrelevant to truth, the irony being that it was the Cartesian algebraization of geometry which inspired Leibniz notion of mathematics. Leibniz' project gained prominence, however, only during the 19th century, when people like Bolzano, DeMorgan, Boole or Grassmann distinguished between "signs of operation" and "signs of quality" (E. Nagel, *Teleology Revisited*, Columbia UP, N.Y. 1979, 183).

Theories began to be considered as realities *sui generis* that must be appreciated and evaluated on their own terms, before one can enter into questions of their meanings or applications.

The first perhaps of taking such a view on mathematical theories was Bolzano. As Cavaillès writes: "Pour la première fois peut-être la science n'est plus considerée comme simple intermédiaire entre l'esprit humain e l'être en soi, dépendant autant de l'un que Le l'autre et n'ayant pas de réalité propre, mais comme un objet sui generis, original dans essence, autonome dans son mouvement" (CAVAILLÈS, 1976, p. 14).

Bolzano therefore pleads for demonstrations or proofs "that show the objective connection and serve not just subjective conviction" (WL Par. 525). He thereby tried to turn proofs into an essential element of mathematical progress proper, instead of just conceiving of them as means of argumentation (Bolzano 1830-1848, 83ff).

This means that the balance between problem solving and theory construction did shift considerably when at the beginning of the 19th century pure mathematics in the modern sense became established. Instead of trying to solve a problem one now asked whether it was solvable at all. There is a difficulty with a merely algorithmic or constructive view of mathematics as it typically characterizes the problem solving approaches: you cannot perform impossibility proofs, which are, however, a kind of birth certificate of modern mathematics.

In his "Rules for the Direction of the Mind" Descartes proposes that the mind requires a fixed method to discover truth. A method is defined as a set of reliable and simple rules (Rule 4) and in his "Discourse on Method" he recommends to be steady and stick with ones decisions, even if some doubts linger, to steadily follow a direction if lost wilderness, for instance, and not to waver around. A little more elaborated algorithm will guide you out of any maze. Mazes can still be solved with the wall follower method, if the entrance and exit to the maze are on the outer walls of the maze. If however, the solver starts inside the maze, it might be on a section disjoint from the exit, and wall followers will continually go around their ring. The Pledge algorithm can solve this problem (see Abelson & diSessa 1980).

The Pledge algorithm, designed to circumvent obstacles, requires an arbitrarily chosen direction to go toward. When an obstacle is met, one hand (say the right hand) is kept along the obstacle while the angles turned are counted. When the solver is facing the original direction again, and the full angular sum of the turns made is 0, the solver leaves the obstacle and continues moving in its original direction. This algorithm allows a person with a compass and a device to count angles to find his way from any point inside to an outer exit of any finite and fair two-dimensional maze, regardless of the initial position of the solver.

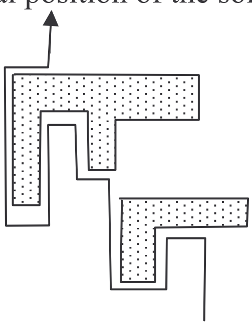

Figure 1

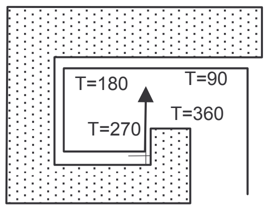

Figure 2

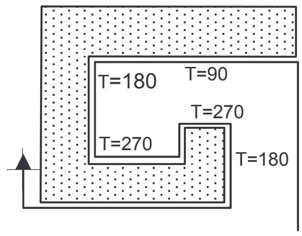

Figure 3

But the person will not get to know the maze. If put back by an evil ghost to the very starting position she will not recognize the spot and has to follow the rules of the algorithm all anew, even if she might have been right besides the exit. What is more serious, however, is that she will not discover whether the maze has in fact an exit at all and may search through the corridors of the labyrinth for ever and ever.

In this manner people in antiquity tried in vain to solve the Delian problem, the problem of the duplication of the cube and quite a number of other problems. Only in 1834 Wantzel showed its impossibility. Descartes had solved the problem using his special compasses for tracing algebraic curves, conics, for example (see Bos 2001), but had never interested himself in (relative) impossibility proofs. The latter marked, however, the development of pure mathematics during the 19th century and its transformation into an analytic science.

When Bolzano and others characterized mathematics as the science of possibility they were promoting an analytical ideal of mathematical knowledge. Mathematicians since that time, rather than trying to construct a mathematical relationship, first asked "whether such a relation is indeed possible", as Abel expressed it in his 1826 memoir *On the Algebraic Resolution of Equations,* in which he presented one of the famous impossibility proofs of modern mathematics. Mathematics, in a sense, became meta-mathematics or became an analysis of the meanings of mathematical concepts and propositions.

Now consider Hilbert`s 10^{th} problem, that is, the *determination of the solvability of a diophantine equation*: Given a diophantine equation with any number of unknown quantities and with rational integral numerical coefficients: To devise a process according to which it can be determined by a finite number of operations whether the equation is solvable in rational integers. "The most direct approach would be to simply test all possible sets of values of the unknown, one after another, until a solution is found. … It is clear that if a solution exists it will be found in a finite number of steps" (M. Davis/R. Hersh, Hilbert`s 10^{th} Problem, *Scientific American*, vol. 229 (1973), 84-91, p. 87).

But if no solution exists for some give equation, something we do not know, however, we are in a maze without exit, and all that will ever be known, is that a solution has not been found yet. And it has been shown by Matyasevich that no mechanical procedure, no algorithm that could be programmed on a computer can be devised to solve Hilbert`s problem.

The present paper pretends to give an exemplary and more detailed presentation of these differences and will suggest that they should be considered as complementary elements of mathematical knowing and learning, because as Gowers observed the division into two different cultures "is not an entirely healthy state".

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