COPING WITH MATHEMATICAL CONTRADICTIONS WITH PEERS

Hagit Sela[1]
University of Maryland, College Park
hagit@umd.edu

Abstract: The study focuses on the process of resolving conflict created by working on mathematical tasks in groups. The participants were exposed to two types of contradiction according to the type of knowledge that they challenged: intuitive knowledge vs. procedural knowledge. Different characteristics of conflict resolution process as related to type of contradiction were found. Changing students' opinions about procedural knowledge seems to be easy with the use of contradicting solutions. However, noticing the distinction between resolving conflict and learning seems to be an important consideration for teaching purposes.

INTRODUCTION

Using contradictions to generate cognitive conflict among learners has been explored in many studies. In most of them, cognitive conflict is considered to be an instructional tool (Zaslavsky, Sela & Leron, 2002; Movshovitz-Hadar, 1993; Dreyfus et al., 1990; Behr & Harel, 1990; Swan, 1983;). Some researchers associate cognitive conflict with the ability to enhance learning due to people's basic need to overcome negative aspects of being in an unbalanced state, and willingness to reach what Piaget (1977) called *equilibrium*. Other researchers emphasize negative aspects of being in cognitive conflict, in that the fragile and intense situation might inhibit learning, in particular for low achievers (Lewis & Dehler, 2000; Smith & Berg, 1987; Duffin & Simpson, 1993). Positive and negative aspects of the impact of cognitive conflict on learning affect researchers' and teachers' positions toward using it as a tool to promote learning.

The goal of the study was to characterize the processes by which learners resolve mathematical contradictions. The study focused on high school students working in groups on tasks involving two types of contradiction. The study relies mainly on two theoretical frameworks related to aspects of students' mathematical learning: cognitive conflict, and group decision-making. These two domains were connected through the tasks with which the participants engaged. In addition to characterizing the process of the conflict resolution, the study aims to find special characteristics of the resolution process related to types of contradiction.

METHODOLOGY AND RESEARCH TASKS

Eight 11th grade students participated in the study. The students learned in a public school in Israel in one of the most selective programs in the country.

The study used four mathematical problems, two of which are presented later. The participants worked on the problems in two settings in a consecutive order. First, each

[1] The study was conducted with Orit Zaslavsky at the Technion – Israel institute of Technology

participant worked on a mathematical problem individually, solving the problem in writing on his or her own. Once all of the students had solved the problem alone, they worked with at least one other student on the same problem. Some were grouped in pairs and others in groups of four (the difference between groups' performance by group size, related to social aspects of group work, is described in Sela & Zaslavsky, 2007). The students were asked to discuss their individual solutions with the other person(s) in their group and to reach agreement on a solution to the same problem. When an agreement was reached, regardless of its mathematical correctness, the group was confronted with an alternative solution, whose conclusion contradicted that of the group. The students were then asked to resolve the contradiction as a group. Thus, the groups followed a two-step decision-making process: first they had to reach an agreement on the solution; second, they had to reach an agreement after being exposed to a solution that contradicted the group's solution. The process of working on the problem in the above setting is called here a *task*.

Contradiction and cognitive conflict

The relationship between contradiction and cognitive conflict is not straightforward. Cognitive conflict is an unpleasant state that people strive to avoid. It is characterized by an unbalanced feeling of tension, frustration, and/or surprise, and an inner drive to resolve the contradiction (Zimmerman & Blom, 1983; Mischel, 1971; Movshovitz-Hadar & Hadass, 1990; Niaz, 1995). One of the ways to create cognitive conflict among learners is by presenting them with a meaningful contradiction. Recognizing a contradiction between one's solution and an alternative solution could bring one to a state of cognitive conflict. Nonetheless, according to the research, there are situations where contradiction exists but students do not experience the unbalanced state of cognitive conflict. In some situations, students ignore the contradiction (Niaz, 1995), and in others, students recognize the contradiction but act in an inconsistent manner (Vinner, 1990; Wilson, 1990). The tasks were chosen according to their high potential for creating cognitive conflict among the participants.

Research Tasks

The study used four mathematical problems that are known to create cognitive conflict among high-school students. Two of the problems are known from literature as contradicting natural conceptions (Tall, 2001), and the others were tested through a pilot study and found to have the potential to cause cognitive conflict for high-school students. In order to infer that students experienced cognitive conflict, we used criteria drawn from literature that characterize the learners' behaviour: anxiety signs such as chewing pens and nails, facial expression and gestures that express frustration, hesitation and doubt, question asking, and/or looking back and forth (Movshovitz-hadar & Hadass, 1990; Zimmerman & Blom, 1983).

The study focused on **two types of contradiction** while trying to characterize the difference between them:

- A contradiction that challenges the students' intuitive knowledge (***CI*** – **C**hallenging **I**ntuitive knowledge): the students face a contradiction because they have an intuitive conception that contradicts the formal approach to the same concept. For this contradiction, the study dealt with the concepts of infinity and infinitesimal. We assumed that before encountering these concepts formally, the students already had some kind of informal concept image about these concepts that would influence his/her way of dealing with them (Tall, 2001).

- A contradiction that challenges the students' formal/procedural knowledge (***CP*** – **C**hallenging **P**rocedural knowledge): the students face a contradiction due to a contrast between a result they get by applying a known procedure and a result someone else got by using another 'method' to solve the same problem. For this contradiction, we used the concept of equivalence in solving equations and inequalities. We relied on research findings according to which most students solve equations using their procedural knowledge without considering the equivalence between the original equation and the equations derived by it by performing operations on both sides of the equation (Fischbein, 1999).

Four problems were used with the goal of eliciting contradictions: two problems elicited a contradiction of type *CI*, and two of type *CP*. All students worked on all four tasks. Each student was interviewed three times during the study. In these interviews, the students were invited to share with the researcher their experiences and feelings regarding the conflict resolution processes.

Qualitative methodologies were applied for the analysis and interpretation of the findings. The transcribed protocols were analysed by means of grounded theory methodology (Goetz & Lecompte, 1984).

PROBLEMS TARGETING THE TWO TYPES OF CONTRADICTIONS

In order to illustrate the two types of contradiction, we describe in detail one problem of each type.

1. A problem targeting *CI* contradiction

> *N* (Natural numbers) and *E* (Even numbers) are the following infinite sets:
>
> ***N*** = {1,2,3,4, 5 , 6, 7 , 8 , 9, 10,11,12,13,14,. . .}
>
> ***E*** = {2,4,6,8,10,12,14,16,18, 20,22,24,26,28,. . .}
>
> Which set has more numbers?

Figure 1: Problem targeting a contradiction that challenges intuitive knowledge

The above is a well-known problem. In order to articulate the contradiction that this problem might produce, we use Tall's (2001) terminology, which distinguishes between *natural conceptions* and *formal conceptions*. According to this distinction, individuals have natural conceptions of infinity that contradict the formal definition of infinity. In Tall's terms, let us think about this problem taking two different points of view, one at a time. A *natural* point of view may lead us to generate the even numbers from the natural numbers by removing the odds from the natural numbers. This creation of taking away half of the natural numbers might yield a conception according to which the set of even numbers has half as many numbers as the set of natural numbers. Moreover, informal experiences always suggest that ‘the whole is greater than the part’, so that the whole set ***N*** of all natural numbers {1, 2, 3, . . .} is "clearly greater" than the part consisting of the set ***E*** of all even numbers {2, 4, 6, . . .}.

But, from a *formal* mathematics point of view, two sets are defined to have 'the same cardinal number' if there exists a one-to-one correspondence between them. In this case, one creates the even numbers from the natural numbers by multiplying each natural by 2. This creation defines a one-to-one correspondence between the two sets and therefore means that there are just as many even numbers as there are natural numbers.

$$\begin{array}{llllll} \boldsymbol{N}: & 1 & 2 & 3\ldots & n & \ldots \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ E: & 2 & 4 & 6\ldots & 2n & \ldots \end{array}$$

Figure 2. A one-to-one correspondence between the natural numbers and the even numbers

The natural point of view is persuasive for students who are familiar with mathematical formality of infinity. Fischbein (1999) and Tall (2001) argue that learners' concepts of infinity usually arise by reflecting on finite experiences and imagining them extended to the infinite. While thinking about this problem, the learner might call on his/her natural conception, or intuitive knowledge, which therefore might generate a contradiction between the natural and the formal conceptions. This contradiction seems to have different attributes than the *CP* contradiction illustrated hereinafter.

2. A problem targeting *CP* contradiction

Solve the following: $\sqrt{x-2} \geq 4-x$

Figure 3: problem targeting a contradiction that challenges procedural knowledge

In order to explain the contradiction that may be produced by this problem, the following section presents a common (wrong) way in which high-school students often solve this inequality, followed by an alternative (correct) way to solve it.

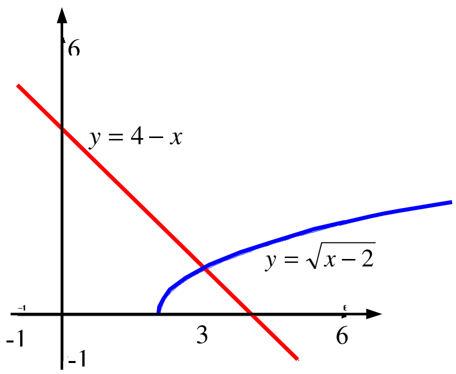
As known from the literature, many students carry out a 'procedure' to solve equations and inequalities,performing operations on both sides without questioning their consequences (Steinberg et al., 1991; Linchevsky & Herscovics, 1996; Esty, 1992). In this problem, students may square both sides of the inequality without thinking about whether the yielded inequality has the same solutions as the original one (see figure 4).

The domain is restricted to $x \geq 2$. Squaring both sides of the inequality yields:

$$\begin{aligned} &\sqrt{x-2} \geq 4-x \\ &x-2 \geq 16-8x+x^2 \\ &x^2-9x+18 \leq 0 \\ &(x-3)(x-6) \leq 0 \\ &3 \leq x \leq 6 \end{aligned}$$

Figure 4: Students' common solution to the problem

As an alternative to the students' solution, we presented them with a graphical representation of the correct solution, which contradicts their solution. We told the students that another student of the same age solved it by using graphs in the following way:



The graphs intersect at x=3. For all values of x greater then 3, the graph of $y = \sqrt{x-2}$ is above the graph of $y = 4 - x$. Thus the solution of the inequality is $x \geq 3$

Figure 5: Alternative (correct) solution which contradicts students' common (wrong) solution

The common 'method' (Figure 4) solution is a close domain $3 \leq x \leq 6$, while the graphical representation suggests an open domain $x \geq 3$. The contradiction between these two solutions may encourage students to rethink their procedural knowledge. In contrast with the contradiction of type *CI*, in dealing with this kind of contradiction, students do not have natural or intuitive conceptions about inequalities of this kind. Their main 'arsenal' is the common procedure of performing the same operation on both sides that they learned in school.

Taking into account the difference between the two types of knowledge students might use while dealing with these two contradictions, one might anticipate that the resolution processes for the two types of contradiction might be different.

FINDINGS

The study sheds light on the characteristics of the process of conflict resolution in general, and highlights major differences between two types of resolution processes depending on the type of knowledge that is 'activated' by the task according to the type of contradiction involved.

Overall view

As shown in the inner column of Figure 6, in 10 cases out of the 12 total cases, the group’s agreed solution to the original problem was wrong (therefore, the contradicting solution presented to them was right). This means that the tasks were efficient in the sense of causing a meaningful contradiction in the desirable direction of learning.

For those 10 cases, there is a difference between the groups' decisions related to the type of contradiction: after being confronted with a contradicting solution, in most of the *CP* cases, the group’s opinion about the solution changed, while in all of the *CI* cases, the group’s opinion about the solution remained the same as it was initially. This difference is elaborated in the following paragraphs.

In all of the *CP* cases mentioned above, the students decided to change their opinion in favour of the right alternative solution. The situation of solving an inequality was familiar to them from school learning. They had tools to check their solution and to compare it to an alternative solution presented to them. They realized that the value $x = 6$ appears as a boundary value in their solution (Figure 4), but not in the alternative solution (Figure 5). This observation made them wonder about the possible reasons of the appearance of the value $x = 6$ in one representation of the solution but not in the other. Substituting different values of x in the original inequality convinced them that the graphical solution represents the solution better than the symbolic solution. As a result, they recalled that there are situation where operating on both sides might add extraneous solutions, such as squaring both sides.

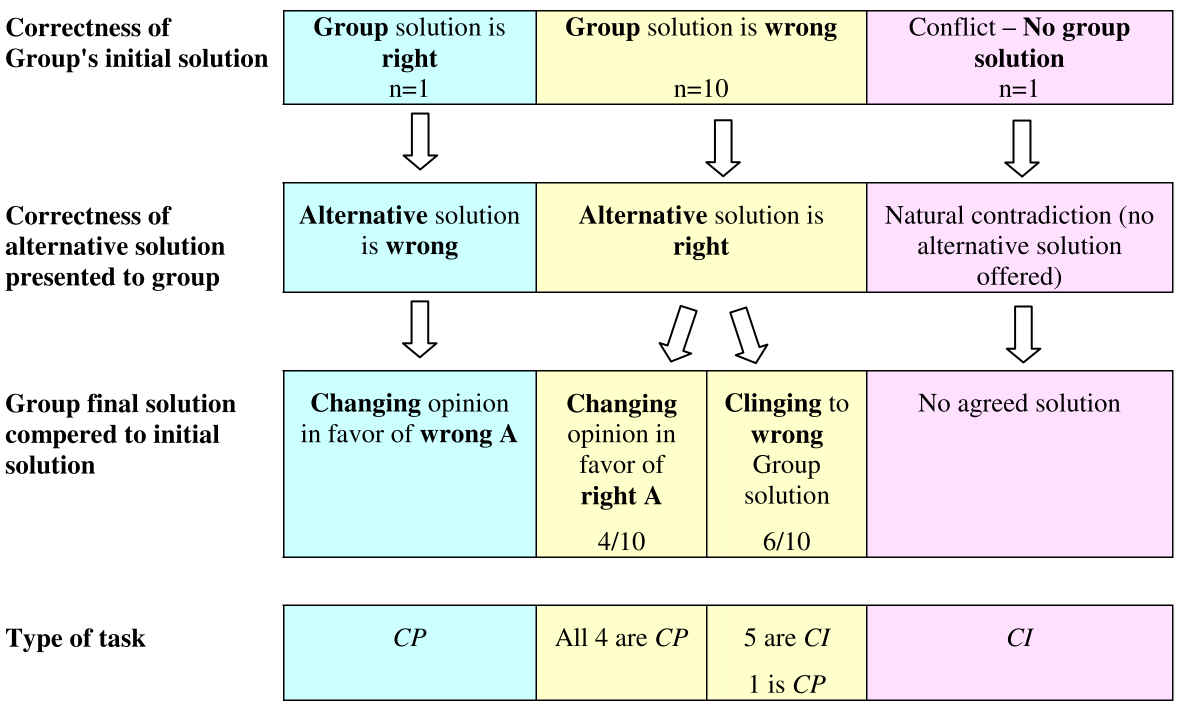


Figure 6: Overview of study group performance (N=12)

The other cases with a wrong initial group solution were *CI* tasks. In these discussions, the students exhibited a behaviour opposite to that in the *CP* tasks: they clung to the wrong agreed solution. This finding is not surprising, taking into account the coerciveness of intuitive knowledge (Fischbein, 1999, 2001). An additional explanation for clinging to their initial solution that is suggested by this study is the students' difficulties in bridging the gap between their initial solution and the contradictory solution. The contradictory solution deals with a correspondence between the natural and the even numbers (Figure 2) and does not relate to the natural conception of 'taking away' the negative numbers from the natural numbers. Although in the CP tasks, the contradictory solution was sufficiently accessible for students to make the shift; in this case, the gap between the two solutions seemed to be too big for them to bridge.

A surprising finding occurred in one of the *CP* cases (left column in Figure 6). The group's initial solution was right, but after being confronted with the wrong alternative solution, they decided to change their mind in favour of the wrong solution. This finding emphasizes the ability to change students' opinions about procedural knowledge and demonstrates the easiness with which negative learning can occur. One can hypothesize that the students in this group were not able to justify their initial solution because their understanding was limited to procedural knowledge only.

Process of conflict resolution in groups

While the previous part dealt with overall findings related to the final resolution, this part deals with the process of conflict resolution itself. In order to characterize the process, the student responses in the discussion were coded and counted. In both types of contradictions (*CI* and *CP*), the students displayed **five types of activities**, which belong to **two types of attempts** to resolve the contradiction:

- *Attempts to reach a sense of certainty*. These attempts are associated with situations in which the participants lack a clear, articulated standpoint about the ways to resolve the contradiction. These attempts are characterised by **three types of activities**:
  1. Re-solving the original problem;
  2. Examining the truth of the solutions;
  3. Comparing the two solutions (group solution and alternative solution).
- *Attempts to support an assertion*. These attempts are associated with situations in which the participants hold a clear standpoint about the way to resolve the contradiction, and therefore make efforts to convince others in the group. These attempts are characterised by **two types of activities**:
  1. Stating the truth or the erroneous of one of the solutions;
  2. Justifying or refuting one of the solutions.

Analysis of the students' responses reveals a difference between the characteristics of the processes of coping with the contradictions as related to the type of contradiction: In *CP* tasks, a similar number of occurrences of the two types of attempts above was found, while in *CI* tasks, a larger use of attempts to *support an assertion* was found. A possible interpretation of this finding is that the students were more open to questioning their knowledge and to examining the truth of their solution in the context of *CP* tasks than in the context of *CI* tasks. The dominance of *supporting an assertion* attempts for resolving *CI* contradictions is related to the certitude of intuitive knowledge (Fischbein, 1999, 2001), which increased the students' sense of confidence in their knowledge. In addition, while intuition is a global conception (Fischbein, 1999), procedural knowledge is a learned 'method' that is built of steps. It seems that it is easier for learners to refine steps in a method than to change a global intuition.

This difference between the characteristics of the processes in relation to the type of contradiction was also found with regard to the **meaning** of the contradiction for the students. The students seemed to be bothered more by the contradictions that arose in *CP* tasks than by the contradictions that arose in *CI* tasks. They explained their wider interest in resolving *CP* tasks by the tasks' relevance to the students' school success. This is in comparison with *CI* tasks, which were treated more like a puzzle that is not particularly relevant for them. This finding might encourage teachers to use contradictions to enhance meaningful discussions of their students' procedural knowledge.

Conflict resolution and learning

After each session of working on two tasks (*CP* and *CI*), the students were interviewed individually. Among other things, they were asked to describe the contradiction and the group resolution of it. Most of the students displayed a poor understanding of the resolution, although they were active in the group decision-making process. Several minutes after the session, they were convinced that the group reached the right answer, but they could not explain it. This finding indicates that resolving a conflict does not ensure learning. A situation in which a learner reaches a point of satisfaction but not an enhanced state of knowledge is an undesirable situation. Teachers have to be aware of this consequence when teaching with contradictions. After experience with cognitive conflict, teachers have to help students with refining and updating their knowledge to a point where they can explain the resolution of the contradiction.

CONCLUSIONS

The study identified typical characteristics of conflict resolution processes in relation to the type of contradiction that created the conflict. These characteristics help us understand a possible reason for students' changing their initial opinions or clinging to them after coping with a contrasting solution.

An ability to bridge the contrasting solutions seems crucial for learners to be able to explore the connections between them. Noticing the commonalities and differences between them enable the students to question their conceptions.

The 'easiness' of changing students' opinions with regards to procedural knowledge illustrates the possible use of counterexample to enhance learning by using contradiction. This conclusion may encourage teachers to use cognitive conflict as a tool for learning in the context of procedural knowledge.

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