FROM DISCOVERIES TO VERIFICATIONS – THEORETICAL FRAMEWORK AND INFERENTIAL ANALYSES OF CLASSROOM INTERACTION

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In a constructivist perspective, each learner has to discover new knowledge on his or her own. The learner creates hypotheses in order to explain 'facts'. Since the discovery of new knowledge alone does not guarantee certainty, the hypothetical knowledge has to be verified. This paper presents a theoretical framework which allows analysing the processes of discovering and verifying knowledge and the coherences between these processes, so that it is possible to analyse the creativity, the plausibility and the needs for proof of a discovery. The theoretical framework is based on the concept of abduction, as described by Charles S. Peirce. He elaborated abduction as the third elementary inference, besides deduction and induction. While the process of forming hypotheses will be described by abduction, the different processes of reasoning and proofing will be described by deduction and induction. The empirical reconstruction of classroom communication has been used to elaborate the theoretical concepts for the research in mathematics education. An example will be used to illustrate the reconstruction.

1. Introduction

For the entire mathematician's experience on the coherence between discoveries and verifications, there is still a lack concerning the theoretical framework which can explain these coherences. Whereas formal logic or the pattern of Toulmin (cf. Krummheuer 1995, Schwarzkopf 2000) have often been used to analyse arguments and proofs, there seem to be no specific patterns in use for analysing processes of discovering knowledge.

The American philosopher Charles Sanders Peirce elaborated abduction as the third elementary inference, besides deduction and induction. While abduction is a conceptual tool for analysing processes of discovering knowledge, induction and deduction and their combinations refer to (empirical and theoretical) processes of verifying knowledge.

In mathematics education the notion of abduction has already been a) considered from a theoretical point of view (cf. Hoffmann 1999) and b) used for the justification of the researcher's interpretations of classroom interaction (e.g. Voigt 1984).

2. Abduction

In the course of his philosophy Peirce offered several different and contradictory patterns and descriptions of abduction. In his later theory he defines the "definitive logical form" as follows:

> "The surprising fact, C, is observed;
> But if A were true, C would be a matter of course,
> Hence, there is reason to suspect that A is true."

Figure 1: Peirce's description of abduction (Peirce, CP 5.189)

Hempel and Oppenheim postulated that every scientific explanation has "to contain at least on general rule" and has "to be inferable from the explanation pure logically" (Stegmüller 1976, p. 452). Altogether we get the following patterns of abduction:

result:	$R(x_0)$	result:	$R(x_0)$
rule:	$\forall i : C(x_i) \Rightarrow R(x_i)$	rule:	$\forall i : C(x_i) \Rightarrow R(x_i)$
case:	$C(x_0)$	case:	$C(x_0)$

Figure 2: General patterns of abduction (left: the cognitive 'flash of genius'; right: abduction as process of making a hypothesis plausible)

Starting with the observing of a surprising fact, we notice a reason which can explain our recognition: A general rule leads to a case, which can explain the facts. If we are aware of the rule the observed fact appears as a specific result of it. This is a necessary condition for the fact getting its 'logical status' as a result. The rule occurs tentatively. It could be that another rule was causal for the given facts. Other cases could be consequences from another rule. Thus abduction is only a hypothetical inference and has nothing to do with "logical inference" in the sense of mathematical formal logic.

The case of an abduction is entirely contained in the rule. So if we are aware of the rule, we are also aware of the case. Therefore we have to differentiate between the abduction as the cognitive process of finding an explanatory rule and case, and the abduction as a communicative act by which we make our hypothesis plausible. The cognitive process starts with only one given premise – the result. This 'flash of genius' can obviously not be interpreted by the scientist.

The existence of only one given fact indicates that we abductively infer not only a case. Also rules can be generated abductively. If a new rule emerges by this inference, Eco (1983, p.

207) calls the abduction “creative”. “Undercoded“ or “overcoded” abductions consist in the explanation of given facts by already known rules. If the association of the known rule is “given automatically or semiautomatically” he classes the abduction “overcoded” (Eco 1983, p. 207). Thus the generation of one discovery can imply a) a new case (all kinds of abduction), b) the relationship between the observed facts and the associated or the generated rule (all kinds of abduction) and c) a new rule (by a creative abduction). As these aspects can only be hypothetical at first place, they have to be verified in the next step.

3. Empirical ways of verification

In his later philosophy Peirce defines abduction, deduction and induction as three steps in the process of inquiry. Having abductively conjectured a hypothesis we deduce necessary and probable experiential consequences from it. These consequences can be tested. The outcome of the test is the result of the following induction by which the hypothesis can be confirmed or refuted (CP 8.209). Two empirical ways can be differentiated: A hypothetical rule emerges from a given set of examples. Another example can be used to verify the rule. Carrier (2000, p 44) calls this empirical way of verification the “bootstrap model”. The (similar) cases, which are used to deduce a necessary consequence, remain within the “boarders” of the generated or associated rule. A discovered arithmetical rule can also be confirmed by testing its geometrical consequences. Carrier (2000, p 44) calls this way of verification the “hypothetic-deductive approach”. Compared with the “bootstrap-model” another rule is used (within the deduction) to draw a consequence from the hypothesis (cf. Meyer 2007, pp. 63).

4. Theoretical ways of verification

The abduction is a hypothetical inference. It is essential for discoveries and also for the explorative phase of finding the idea for a proof. Empirical ways can only make a hypothesis more plausible. Thus Mathematicians often favour deductive proofs. So a theoretical way of verification consists in proving the hypothetical rule and/or the hypothetical case of the abduction deductively. The rule of the abduction becomes the result of a following (chain of) deduction(s). But how do we find the proof? Up to now we have only one given premise: The rule of the abduction as the result of the deduction. We have to ‘step back’ to find a rule (of the deduction) which can be used to infer the rule (of the abduction). Accordingly we are in need of another abduction. Thus finding the idea of a proof also requires abductions.

There is a problem concerning the verification of the abductively conjectured coherence of the result and the rule of an abduction. This coherence can not be proven deductively. It can not precluded that another rule was causal for the observed facts. Empirical ways of verification are not special for empirical science. They are also important for the learning of mathematics. So Jahnke (2007, p. 79; cf. Hanna und Jahnke 1996) centred his article "[…] around the idea that inventing hypotheses and testing their consequences is more productive for the understanding of the epistemological nature of proof than forming elaborate chains of deduction." The logical structures of the empirical ways of verification corroborate this idea and give an insight in the coherences between empirical and theoretical ways of verification. The verification of knowledge by the hypothetical-deductive approach can be compared with a deductive proof of the hypothesis: Both ways of verification are in need of another (abductively conjectured) rule. While the rule of the proof has to precede the abduction logically, the rule of the deduction within the hypothetical-deductive approach succeeds the abduction – not only temporally but also logically.

5. An example of the analyses of classroom communication

In order to show the use of the presented theoretical framework, we will now consider a short example how the framework can be activated for analysing real classroom interaction.

The scene will be interpreted from an ethnomethodological and interactional point of view. It emerged from the first lesson of a classroom experiment in a grade 10 (students aged from 16 to 17 years). The lesson was dedicated to the introduction of the power functions. Students' preknowledge comprised the parabola and the calculation with x to the power of n. Now the students should complete "$x \mapsto$ " for the dashed function on their working sheet (fig. 3).

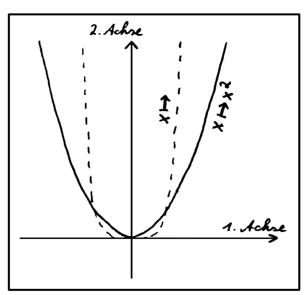

Figure 3: Given functions on the working sheet ("Achse" is the German word for "axis")

The students first could not solve the task. The teacher offers them the following suggestions: $x \mapsto 10 \cdot x^2$, $x \mapsto 0{,}1 \cdot x^2$, $x \mapsto x^2 + 10$, $x \mapsto x^5$, $x \mapsto 2^x$. Now we take a closer look on Eva's statement:

> maybe it could be something with ehm, for my sake x^4 or like that because, eh the, the eh (*soft spoken*) no it is nonsense x^4 (*teacher writes* $x \mapsto x^4$ *on the blackboard, Eva speaks up*) as x^4 because eh, if you insert, than eh, a negative number for x, eh there will come something positive out of eh, of the, of the function as ehm, on the, x-axis eh so if you are now right eh from the point of origin, (*teacher point at the point of origin*) than there still has to be a ehm, still ehm, thus still a positive eh result so that you actually can get high elsewise you have to go deeper if eh if there, is x^5 for my sake you can, so minus times minus is plus, than another time minus times minus is again plus and than, another time minus times minus is yeah eh minus, so you would land in the range of the negative numbers and than the graph would drop down.

I would interpret this scene as follows: Eva assumes that the dashed graph can be the graph of the function $x \mapsto x^4$. She refers her hypothesis to the course of the graph[1]. So we get the following abduction:

result:	The graph runs upwards for negative values of x.
rule:	If the term has the form x^n (n even), than the corresponding graph will run upwards.
case:	The term of the graph could be "something with" x^4.

Figure 4: The abduction concerning Eva's statement

As power functions with exponents greater than 2 had not been discussed before, this scene is an example how students set up new rules by a creative abduction. Eva did not mention the rule of her abduction explicitly, though she used it implicitly. Her following argument can be separated in two parts:

The first part can be interpreted as a deductive proof of the implicit rule of the abduction: Starting with $x \mapsto x^n$ (n even) Eva deduces by multiplying x successively with itself, that the product has to be positive, if x is any negative number. In another

[1] By the way: Eva mentioned "right eh from the point of origin". In the following course of the lesson she is going to correct herself.

step she infers that the corresponding graph has to run upwards. In detail three deductions can be reconstructed in this section (cf. Meyer 2007, pp. 197). Nearly everything of this proof remains implicit. That Eva is aware of it can be interpreted by the second part of her argument. Here she expresses a nearly analogous argument for odd exponents in $x \mapsto x^n$.

The function of her argument in the second part can be interpreted in different ways as a proof of the case of the abduction. Every possibility is in need of the assumption that the term in question is of the form x^n: 1. The argument can be regarded as a proof by contradiction, when we assume that Eva implicitly compares the deduced course of the graph with the given graph. 2. Eva's statement could be a proof by contradiction for the converse direction of the implication. Thus the rule of abduction turns into a statement of equivalence and the case of the abduction can be inferred from the result of the abduction deductively. 3. As she infers the consequences for n being even or odd, we can assume that Eva does a complete case differentiation.

The reconstruction shows that the process of discovering knowledge starts with only one given premise. In this scene the students first could not use this premise to execute their abduction. With the mention of $x \mapsto x^5$ the teacher gave them a hint, which Eva grasped. The hint of the teachers can be interpreted as a reduction of the semantic fields between the result and the case of the abduction. Thus a creative abduction can be forced from outside. Even though the hint of the teacher has been given before, the concept of abduction shows that the students have to notice the coherence between the case and the result.

Within the reconstruction it has not been possible to determine the relevance of the second part of Eva's argument for the proof the case of her abduction. Comparable problems emerged within the reconstruction of other scenes. The reconstruction shows that it is not only problematic to differentiate between a) different theoretical ways of verification and b) theoretical and empirical ways of verification. The decision can depend on implicit assumptions of the student.

6. Future prospects

In this article I described the inference abduction, its relations to deduction and induction and gave a short insight into the analyses of classroom interaction. Together with J. Voigt I have also used the patterns to analyse mathematical school books. One result of the analyses is the following: In a school book below the picture (fig. 5) there is the task: “Formulate a rule which helps you – only by calculating the sum of digits of a number – to make a statement on the divisibility of this number by 3 or 9”.

$$\begin{aligned} 4752 &= 4 \cdot 1000 + 7 \cdot 100 + 5 \cdot 10 + 2 \cdot 1 \\ &= 4 \cdot (999 + 1) + 7 \cdot (99 + 1) + 5 \cdot (9 + 1) + 2 \cdot 1 \\ &= 4 \cdot 999 + 4 + 7 \cdot 99 + 7 + 5 \cdot 9 + 5 + 2 \\ &= 4 \cdot 999 + 7 \cdot 99 + 5 \cdot 9 + (\underline{4 + 7 + 5 + 2}) \end{aligned}$$

Figure 5: Excerpt from a mathematical school book (Esper and Schornstein 2006, p. 137)

It is still a problem that students often do not see the idea of a proof. Tasks like this may be an alternative: In order to find a rule the students are only confronted with the calculation in the figure. So this could be taken as the result of the (creative) abduction by which the students should generate the required rule. Every step in figure 5 can be reconstructed as a deduction. Thus the concrete result is getting inferred by a chain of deductions. If these deductions are going to be generalized, it could be possible to proof the abductively conjectured rule with these deductions. In other words: The task is an example of what we call “discovery with a latent idea of proof”. The idea is latent because the students have to notice it. But this is of course not self-evident.

7. Final remarks

Peirce’s differentiation between abduction and induction allows a detailed analysis of processes of discoveries, verifications and their coherences. The abduction gives an insight in the processes of discovering knowledge and also in the processes of finding an idea for a proof. The combinations of the inferences give an insight in the processes of verifying hypothetical knowledge.

Let us go back to the claims from the beginning: The representation of an abduction enables the recognition of their rationality and plausibility. Eco’s types of abductions show the creativity of discoveries and also their needs of proofs. The reconstruction of verifications as being empirical or theoretical allows grasping their conclusiveness and their potential persuasive power. However, the persuasive power is always subjectively. E.g.: In the following, here not documented course of the lesson, the students showed that they were not convinced by Eva’s solution. They asked for examples (see also Fishbein 1982, p. 16). The analyses show that the "power of the best argument" must not depend on logical conclusiveness.

8. Literature

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