ANALYSIS OF THE LEARNING PROCESS OF THE CONCEPTS OF FUNCTION AND EXTREME POINTS OF FUNCTIONS ON ECONOMICS STUDENTS

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A research into the field of mathematical education, concerning the difficulties in the teaching-learning process of the concepts of function and extreme points of functions on economics undergraduate students at the Universidad Veracruzana in Mexico, is reported. The paper takes as its theoretical framework the main contributions of the research programme known as Advanced Mathematical Thinking. Our conclusions agree with those of previous studies. The authors detect difficulties regarding interpretation and construction tasks related to the concept of function, then state that the student's cognitive structure is more related to some characteristics of the function than to the concept itself. The concept of extreme point of a function is explained by the students, in some cases, by means of the relative position of this point inside a neighborhood of values and on the basis of graphical representations.

INTRODUCTION

This paper reports a research about the difficulties in the teaching-learning process of the concepts of function and of extreme point of a function on economics undergraduate students. The authors approach the subject matter of functions and function graphs, and use as a theoretical framework the main contributions of the research programme known as Advanced Mathematical Thinking, particularly the distinction between concept definition and concept image (Tall & Vinner, 1981), and also other theoretical constructions concerning the cognitive processes of learning. The purpose was to analyze, from this viewpoint, the teaching-learning process of these concepts. The authors attempted to evaluate the learning process on both a class who attended the traditional *Calculus I* course, and a class who only studied intuitive ideas about these concepts.

During the first stage of this research, a diagnostic survey was carried out looking for difficulties in the understanding of these concepts after the course of *Calculus I*. The subject of study was a class conformed by 36 students, and a sample of 10 students for the case study. The authors intended to identify difficulties regarding the relationship between the concept of extreme point, the concept of function, and the different representations of the latter. During the second stage, the authors used a didactic innovation titled “Functions, its Representations, and Extreme Points of Functions” on a group formed by students who had not attended *Calculus I*. The authors have this second group study (58 students) only the intuitive ideas of function and of extreme point of a function, so the authors could analyze the learning process within a different teaching framework.

THE CURRICULA: A FRAMEWORK OF REFERENCE FOR THE RESEARCH

Students in Mexico go to University after they have graduated from secondary school (junior high) and from preparatory school (middle school), which curricula are prescribed by the Mexican Ministry of Education (Secretaría de Educación Pública [SEP]). Regarding mathematics:

> Instruction in secondary and preparatory schools has as its general purpose to develop operational, communicative and discovery skills on students, who must get the ability to:
>
> - Acquire, by means of problem solving, self-confidence and competences for the use of basic techniques and procedures.
> - Recognize and analyze different aspects of a problem.
> - Elaborate, communicate and validate conjectures.
> - Choose or adapt the suitable strategy for problem solving.
> - Communicate strategies, procedures and results clearly and succinctly.
> - Predict and generalize results. (SEP, n. d., Trans. Méndez).

In practice, nevertheless, the educational system demands that students acquire knowledge, most of the time by direct transfer from a teacher. The teaching process is carried out through a direct speech by a teacher, followed by the students’ attempt to solve routine exercises from the textbook. In this way, the process does not motivate a restructuring of the student’s ideas, in the sense of Azcárate (1992), and so, it does not raise a cognitive conflict between what the student knows and those new situations s/he faces.

On the other hand, the instruction in the *Calculus I* course for economics undergraduates initiates with a presentation of the mathematical theory of functions and the finding of extreme points of functions, and culminates with routine exercises allegedly related to the core of economics. The teacher’s explanations follow a logical order, but keeping an intuitive level (using here the term *intuitive* as an antonym of *rigorous*, like in Tall [1985]), and trying that students solve exercises similar to those in the textbook. However, a previous empirical study (Cuesta, 2007) indicates that students have serious difficulties acquiring, as the syllabus demands, the knowledge about the concept of function in order to “enable them to subsequently acquire competences to pose and solve elementary optimization problems in economics” (Méndez, Bustamante & Cuesta, 2004, p. 2, Trans. Méndez).

It is easy to realize that, within this teaching environment, students do not assimilate the whole concept of function, which would enable them to organize their mental processes (Tall, 1985) and to construct meaningful knowledge, both in mathematics and in economics. Research into the field of mathematical education (Dreyfus & Eisenberg, 1982; Tall, 1989; Artigue, 1990; Leinhardt, Zaslavsky & Stein, 1990) has shown the existence of difficulties in the learning of mathematical concepts; some of these are caused by a traditional teaching method (Dreyfus, 1991; Artigue, 1995), in which the teacher evaluates the students' acquisition of knowledge in terms of algorithmic and algebraic skills.

It is true that the difficulties are strongly related to the student's personal experience (Artigue, 1990); but both difficulties and student's experience are a consequence of a teaching environment which does not motivate associations between concept images and concept definitions (Vinner, 1991). This fact results from memory based learning and from external representations of concepts (Dreyfus, 1991) which do not contribute to the learning process. To a large extent, the student's conceptions are characterized (Artigue, 2003) as the authors describe next:

- The students believe that they should only learn the contents of an examination.
- The student's conceptions remain fuzzy, incoherent and poorly adapted to the requirements of the calculus' world.
- Most of the students believe that, in order to succeed in calculus courses, is better to act mechanically, without attempting to achieve any comprehension.

In brief, reality contradicts the essence of teaching as Azcárate (1997) conceived it: "teaching-learning process consists, to a large extent, in the gradual sharing, between teacher and students, of the concept images of those mathematical notions which are the study matter" (p. 29, Trans. Méndez). Different from what actually occurs in the classroom, it is possible to consider an inquiry-oriented teaching (Herbel-Eisenmann, Lubienski & Id-Deen, 2006), to elaborate teaching strategies, and to identify new teaching resources and methods, which aim to stimulate the learning process, and to allow the formulation and corroboration of conjectures. Therefore, it is necessary to motivate the restructuring of the students' ideas (Azcárate, 1992), so they will be able to construct new concept images and to change their comprehension of the real and mathematical worlds.

The didactic innovation is a new learning environment (Cuesta, 2007), where the objectives are specifically designed in order to achieve a goal. Such objectives are clearer and, different from *Calculus I* course, they take advantage of the graphical environment. This constitutes a more qualitative way to introduce the concepts of function and of extreme points of functions, motivating the genesis of intuitive ideas from real world situations, some of them related to the students' knowledge on economics.

DATA GATHERING

Our first interest was to determine the difficulties in the learning of those concepts, but also the sources of knowledge used by students when answering questions about functions and extreme points of functions. To this end, a written test (exam 1) was applied to students attending *Calculus I* (group 1); after that, the authors interviewed a sample of 10 students (sample 1), aiming to study in depth the students' argumentations for the answers given to the questions in exam 1. The authors expected to find the specific difficulties through guided conversations with the students.

The second student group (group 2) was formed by those who participated in the implementation of the didactic innovation without having attended the traditional *Calculus I* course. This innovation consists of three sequences of teaching-learning activities: (1) reading and interpretation of graphs; (2) analysis of varying phenomena; (3) the concept of function, and characteristics of functions. The objectives were for the students to:

- Become familiar with the kind of information contained in graphs, and its significance in every situation under study.
- Gain an understanding of the ideas of variable and of functional dependence, based on the analysis of varying phenomena.
- Become able to identify, in a variety of situations, which would be the independent variable and which would be the dependent variable; also to choose suitable units of measurement for either one.
- Become familiar with the function concept, based on different languages: verbal, numerical, graphical and algebraic.
- Become familiar with the intuitive idea of extreme point of a function.

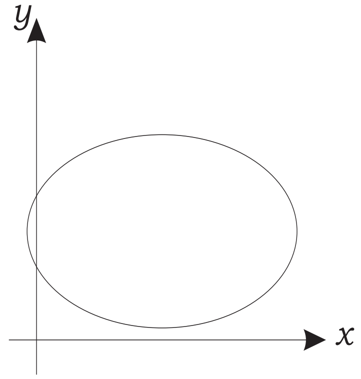
In order to identify the difficulties in the second group, the authors used various data gathering instruments: direct observation inside the classroom, case study, interviews and written tests. At the end, the authors applied a written test (exam 2) to the group. Based on the answers to this written test, the authors interpreted the student's conceptions about function and extreme points of functions, but also analyzed the difficulties regarding interpretation and construction tasks related to the concept of function. For the case study, the authors drew a sample (sample 2) consisting of six students, aiming to analyze their argumentations when answering the questions from exam 2.

In this manner, the methodology used for the research was qualitative and descriptive.

To this paper ends, the authors consider only some of the questions set to both groups, specifically:

1. What is a function? Explain your answer.

2. Does the following graph represent any univariate function? Explain your answer.



3. What do you mean by local maximum or minimum of a function?
4. For group 1: A farmer has 320 meters of fence to enclose a rectangular field. How should he use the fence if he wishes the enclosed area be as big as possible?
   For group 2: The adjacent sides of a rectangle add up to 15 cm. Analyze how the rectangle's area vary when its sides length vary.

Question 4 is intended to analyze how students interpret the geometric context, and how they translate between different representations of a function: from the geometric context to a table of values, to a graph, to the algebraic expression.

OUTCOME ANALYSIS

After analyzing the students' argumentations in handwritten answers to the written tests and in the interviews' transcripts, the authors found several sources of knowledge which the authors describe within the following paragraphs.

Sources of knowledge related to the function concept

The authors found that sources of knowledge related to the function concept are:

Rule: The student recognized the existence of a rule which relates the values of one variable quantity to the values of another variable quantity, in such a way that the value of the second variable quantity is uniquely determined by the value of the first.

Correspondence relation: The student's argument was based on the idea of correspondence between the elements of two sets: domain and codomain, in such a way that corresponding to each value in the domain there is one and only one value in the codomain.

Dependence between variables: The student only referred to the dependence of one variable on the other.

Symbolic expression: The student showed a rudimentary interpretation of the concept. S/he recognized the function as a mathematical formula, an algebraic expression or an equation. S/he declared that this formula is useful to solve something or to obtain any outcomes.

In Table 1, absolute frequencies for the different sources of knowledge are reported, for each sample.

Argumentation	Sample 1 count	Sample 2 count
Rule	1	4
Correspondence relation	1	0
Dependence between variables	3	0
Symbolic expression	5	2
Sum total	**10**	**6**

Table 1: Answers to question 1

Note that the answers from students in group 1 (those who attended the traditional *Calculus I* course) were farther from the concept definition than those from students in group 2. Some students, indeed, associated the function with an equation or an algebraic representation, e. g.: "I don't know what a function is, but if I write it down, I'm able to identify that it is a function". Moreover, these students conceived the algebraic and graphical representations as separate things, e. g.: "A function is an algebraic expression, and we must assign values (sic) if we want to draw its graph".

A second difficulty regards question 2. Those students who used the rule argumentation could not explain why the graph does not represent a function, e. g.: "It is not a function because there are two axes (sic) corresponding to each axis (sic)"; "If it were a function, then a cycle (sic) took place because there are two *y* values corresponding to each *x* value". Other students evoked the examples studied in the classroom, e. g.: "I don't know. We studied several graphs in classes, but none of this kind". Other ones based their arguments only on the image of the graph, but they did not understand the meaning of the concepts neither were capable of indicating which variables were involved.

Sources of knowledge related to the concept of local maximum (minimum)

The authors found that sources of knowledge regarding the concept of local maximum (minimum) are:

<u>Changes in the behavior of a function:</u> Some students explained the local maximum (or minimum) on the basis of a change in the growth behavior of the function, e. g.: "The maximum is obtained when the function is increasing but beyond that point and then it starts decreasing".

<u>Value within a neighborhood:</u> Some other students explained the local maximum (or minimum) as a point on the graph, such that values within a neighborhood are all smaller (greater), e. g.: "The minimum is the smaller value within an interval, and all the values before and after must be greater than it (sic)".

Association with the idea of height: The existence of local maximum (or minimum) was also explained in terms of the higher (or lower) points in the graph, e. g.: "The maximum is the higher point (sic)".

Explanations based only on images: Some students pointed at the maximum (or minimum) on the graph, but were not able to explain the relation between the considered variables.

In Table 2, absolute frequencies for the different sources of knowledge are reported, for each sample.

Argumentation	**Sample 1 count**	**Sample 2 count**
Changes in the behavior of a function	1	2
Value within a neighborhood	2	3
Association with the idea of height	4	1
Explanations based only on images	3	0
Sum total	**10**	**6**

Table 2: Answers to question 3

Answers from students in sample 1 were, at best, related to procedural aspects of the local maximum (or minimum) concept, e. g.: "The maximum and minimum points are those that we find using the first and second derivatives".

Many difficulties appeared concerning construction tasks related to the concepts of maximum and minimum (question 4). For most of the students in sample 1 a conflict arose when they attempted to solve question 4. They were not able to answer it because they ignored the relation existing between the rectangle's perimeter and area. When they were questioned on their difficulties, they answered the following: "The problem states that the farmer has 320 meters of fence, but it doesn't state which are the rectangle's length and width". Students have a fuzzy idea about the rectangle's perimeter, but they know well the formula to calculate such area: "$A = b \cdot h$".

For sample 2, half of the students were able to raise the function involved in question 4. For the rest of the cases, the difficulty arose because they were not able to translate the geometrical context into the language of functions; they were not able to translate the phrase "the adjacent sides of a rectangle add up to 15 cm." into a table of values, or a graph or an algebraic equation.

Two are the main constraints when conceiving and constructing the different ways of representing a function: (1) the lack of knowledge; (2) pre-existing misconceptions. In addition, many students were not able to understand the verbal language, which is probably a consequence of the teaching-learning process based on the repetition of routine exercises that were meaningless for students.

In brief, the authors found the following difficulties:

- Explanations based only on images.
- Misunderstanding of the dependence relation between variables.
- Misunderstanding of functional behavior.
- Misunderstanding of algebraic language in a geometrical framework.

CONCLUSIONS

The conclusions of this research are founded on both the characteristics of the Mexican educational system and on the methodology of the authors. For the purposes of these conclusions, the references found in parenthesis only mean that our conclusions are similar to the authors mentioned.

Conclusions concerning the function concept

The function concept was generally associated with the existence of a dependence relation, but the rule which describes dependency was often misunderstood. Many students were not able to establish the relation between the involved variables (Even & Bruckheimer, 1998), while other ones succeeded in defining the function concept but failed in deciding if a graph represents a function or not (Leinhardt et al., 1990). The misconceptions about a function domain and codomain led students to misunderstandings about the functional rule (Markovits, Eylon & Bruckheimer, 1986).

In order to answer questions or to solve problems, students evoked memories or mental prototypes from issues explained in the classroom (Tall & Bakar, 1992; Fabra & Deulofeu, 2000). The student's cognitive structure was more related to some characteristics of the function than to the concept itself (Azcárate, 1995), especially when these characteristics were viewed through the graphical representation. Some difficulties were closely related to student's prior experiences (Artigue, 1990), but personal experience was based on the issues explained in the classroom, and s/he was not able to use those concepts in a flexible manner when posing and solving problems (Dreyfus, 1991).

Interpretation and construction tasks related to the function concept were affected by the mixed effects of the prior meanings of the concept and of the knowledge about the environments in which those tasks must be accomplished (Janvier, 1987). The authors confirmed that:

(1) The students' knowledge about geometry issues was very poor. Many students faced conflicts when they were asked to construct different representations of a function from a geometrical context.

(2) Students did not show skills to set relations between different representations of a function, such as models, charts, tables of values, graphs, verbal and written language, and algebraic expressions (Lesh, Post & Behr, 1987).

(3) Many students were not capable to work with representations of the concept of function different from the algebraic one (Dreyfus, 1991).

These findings are consistent with the difficulty stated by Artigue (1990) about the rupture between algebra and calculus. They are also consistent with the results from Steinbring's study (1993), which states that problems emerge from the representation contexts; the authors found in this study that the main difficulty arose from the poor level of comprehension of the algebraic language within a geometrical context.

A remarkable fact is that students who attended the traditional calculus course showed difficulties related to the understanding of the function concept. The authors agree with Markovits et al. (1986), Leinhardt et al. (1990), Steinbring (1993), Even and Bruckheimer (1998), and emphasize that:

- Students did not understand the functional dependence of one variable on the other.
- Students did not identify the function domain and codomain, neither the relation between these two.
- Students tended to assume linearity (constant change rates) as a characteristic feature of functions.
- Despite the insistence in algebraic representations, students showed ignorance about its meaning.

Conclusions concerning the local maximum and minimum concepts

At best, students explained the maximum and minimum concepts on the basis of changes in the function growth behavior or by means of the relative position of the extreme point within a neighborhood of values. However, the concept image evoked when explaining their answers was the graphical representation (Tall, 1991), in which the student feels more comfortable to communicate her/his ideas. In this regard, the difficulty became evident when the authors observed that almost all students who used the graphical representation to explain maxima and minima did not mention the function concept, neither the idea of change in its behavior. Students were not capable to establish the relation between two representation systems (Lesh et al., 1987): the graphical representation and the verbal description.

By the time they had initiated their university courses, many students had a concept image (Tall & Vinner, 1981) consisting of the association of the idea of extreme point with the idea of height. It is remarkable that this concept image remained unchanged after the traditional *Calculus I* course. The authors corroborated that many difficulties in the learning of the concept of extreme point arise from the student's previous knowledge of the function concept, to the extent that the definition of function itself becomes an obstacle for the learning of the concept of extreme point.

Conclusions concerning the didactic innovation

The didactic innovation is the first attempt to improve a teaching methodology, nowadays focused on the mere knowledge transferal from teacher to students. Our own teaching experience and the empirical evidence (Cuesta, 2007) demonstrate that students have serious difficulties to acquire, as the *Calculus I* syllabus demands, the knowledge about the concept of function which would make them able to subsequently learn, use and apply the concepts of limit, derivative, and integral; also acquiring competences to pose and solve elementary optimization problems in economics.

According to this, the innovation is a new learning environment where it is possible to take major advantages of the student's prior knowledge, adding mathematical knowledge to the student's cognitive development. Resulting from the use of this innovation, students improved their comprehension of concepts compared to students who attended the traditional course.

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