

BRIDGING THE GAP BETWEEN MATHEMATICS AND PHYSICS

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June 14, 2008

Abstract

The Vector Calculus Bridge Project is an NSF-funded effort, begun in 2001, to understand the differences in perspective between mathematicians and physicists and why these differences cause transition problems for students. As part of this project, we designed and classroom-tested curricular materials at Oregon State University, and also developed resources for mathematics faculty to help them appreciate the needs of their physical science and engineering students. These resources include a series of papers emphasizing the importance of a coherent, geometric approach to the material, group activities and an instructor's guide focused on student development of geometric reasoning, and a series of faculty development workshops.

We present here an overview of the Bridge Project, followed by a case study of one of the early beta test sites, documenting growth in students' geometric understanding and problem solving ability. These skills are essential for a successful transition from lower-division mathematics courses to upper-division physics courses. Our primary conclusion is that emphasizing the geometric meaning of vector calculus helps students to make sense out of what they are learning in this course.

1 Introduction

The Vector Calculus Bridge Project [1] grew out of the frustrations of two of the authors (TD & CAM), a husband-and-wife team of mathematical physicists, over the apparent mismatch between the material taught by mathematicians and the skills needed in subsequent physics classes. Extensive discussions eventually suggested two fundamental differences between these two disciplines. First of all, mathematicians and physicists speak different languages, but with the same vocabulary. This makes superficial conversations about course content, such as comparing syllabi, ineffective. Even more importantly, mathematicians tend to emphasize abstract symbolic manipulation, whereas physicists emphasize problem-solving in context. We like to summarize this difference by saying that mathematicians do algebra, while physicists do geometry.

The Bridge Project strives to bridge this gap by emphasizing geometric reasoning skills. By restructuring the course around a single geometric idea, and emphasizing problem-solving strategies more than algorithms, students not only begin to develop the skills they need later, but also retain more of the traditional content.

Given the absence of materials supporting this approach to vector calculus, we wrote our own. These materials consist primarily of guided activities for the student, together with an extensive instructor's guide.

2 An Example

Here's our favorite example:

Suppose $T(x, y) = k(x^2 + y^2)$. What is $T(r, \theta)$?

We often ask this question of mathematicians and other scientists. Some mathematicians say " $k(r^2 + \theta^2)$ ". Many mathematicians refuse to answer, claiming that the question is ambiguous. Everyone else, including some mathematicians, says " kr^2 ". One colleague, who holds a split appointment in mathematics and physics, simply laughed, then asked which hat he should wear when answering the question. What's going on here?

Just as x and y have standard meanings as rectangular coordinates, r and θ are the standard labels for polar coordinates, with r denoting the distance from the origin and θ the angle counterclockwise from the positive x -axis. So $r^2 = x^2 + y^2$, and if you express $k(x^2 + y^2)$ in polar coordinates you get kr^2 .

But wait a minute; that wasn't the question! $T(x, y)$ is a function of two variables, x and y . It doesn't matter what you call them; r and θ are as good as any other names. So replace x by r and y by θ ; the answer is clearly $k(r^2 + \theta^2)$.

This is of course exactly what mathematicians teach their students about functions, so it is especially noteworthy that many mathematicians nonetheless give the polar coordinate answer.

What is the point? That the mathematics we teach tends to be about formal manipulation of symbols according to well-defined rules, whereas the mathematics we use always

has a context. In this example, many mathematicians recognize the context and use this additional information when answering the question. Nonmathematicians have to do this with every problem, but this skill is rarely taught. Students often express their inability to exploit the context with the words, “I just don’t know how to get started.”

And yes, a physicist really will write $T(x, y) = k(x^2 + y^2)$ for, say, the temperature on a rectangular metal slab, and $T(r, \theta) = kr^2$ for the same temperature in polar coordinates, even though the mathematician would argue that the symbol T is being used for two different functions. This is not sloppy mathematics on the part of the physicist; it’s a different language. T is the temperature, a physical quantity which is a function of position; the letters which follow merely indicate which coordinate system one is using to label the position. This can be rigorously translated into the differential geometer’s notion of a scalar field, or phrased more informally as:

Science is about physical quantities, not about functions.

So not only do other scientists speak a different language, they use the same vocabulary!

3 The Bridge Project

We started the *Vector Calculus Bridge Project* [1] in order to bridge the language gap at the level of second-year calculus [2]. The creation of this project was strongly influenced by our work with the *Paradigms in Physics Project* [3], a major NSF-supported reform of the upper-division physics major, which in turn was motivated in part by trying to help students make the transition from lower-division mathematics to upper-division physics.

We initially compared syllabi between a vector calculus class and the junior-level physics class which “reviews” similar material. After getting nowhere — the list of math topics was nearly identical to the list of physics topics — we finally compared the actual content. We eventually realized that we could sum up the differences in a single sentence:

Mathematicians teach algebra; physicists do geometry.

Physicists (and other scientists) tend to reason geometrically, rather than (or, more precisely, in addition to) “mere” symbol pushing. The temperature is a physical quantity, whose representation as a function is secondary to the fact that the temperature here is, say, 70° , while over there it might be 75° .

Returning to the example, the physicist “sees” the temperature as living on the (2-dimensional) metal slab itself, perhaps by associating a color with different temperature ranges, or equivalently in terms of isothermal contour lines — something like a topographic map. This is very different from the graph of the corresponding function, which in this case is a (3-dimensional) paraboloid. This difference in viewpoint is especially important since physical quantities usually depend on three spatial variables; the corresponding graphs would therefore require four dimensions.

By emphasizing geometry, we have been able to unify our traditional vector calculus class around a single idea (the infinitesimal displacement vector $d\vec{r}$; see [4]). Our other main

ingredient is the use of small group activities with open-ended problems, similar but not identical to the MathExcel [5] and Peer Led Team Learning [6] programs. Not only do we now cover more material in more depth than before, but students seem to be coming away with a deeper (and yes, more geometric) understanding.

The primary Bridge Project materials consist of a series of guided group activities, together with an extensive Instructor’s Guide. We have spoken and written extensively about our work [2, 4, 7, 8, 9, 10, 11, 12], and have a textbook in preparation. Many of these materials are available on the project website [1].

4 Evaluation

In order to investigate the impact of our materials on students, we evaluated both students’ attitudes toward the materials and the nature of their conceptual understanding of the materials. Fifteen students from three different classes were involved in the study. Four of these students were from a “traditional” vector calculus course with three lectures and one recitation per week. This recitation, led by a graduate teaching assistant, was designed to answer students’ questions on the homework or lecture, and to give periodic quizzes. Four further students were in another section of the same course at the same institution, but with a different instructor and teaching assistant. This instructor used Bridge Project materials, turning the recitation into a “lab” with group activities, and made an effort to connect the lectures to the labs. The remaining seven students in the study took a course team taught by the authors of the materials, but at another institution. All participants were volunteers.

Student volunteers participated in two 45–60 minute task-based interviews — one during the first week of their respective vector calculus course and one during the last two weeks of their course. The purpose of the first interview was to give students a task that would allow us to determine in some way their mathematical and visualization abilities at the start of the course, and to determine if our student participant groups were comparable. In the interview we asked students about their educational backgrounds in mathematics; we asked them about their hobbies; and we asked them the following question:

Construct a graph of $x^2 + y^2 + z = 50$.

There were students in each group who were able to graph this equation, and some in each group who were not able to graph the equation. Their methods varied, but among the three groups there was no group which, in general, seemed more able than any other group. Furthermore, their mathematics backgrounds were comparable. We thus concluded that none of the three groups seemed to be entering the course with more “ability” than the others.

The primary purpose of the second interview was to evaluate the effectiveness of the new materials in helping students have a deeper and more diverse understanding of some of the concepts of vector calculus. Most of the second interview was devoted to students engaging in the following task:

Evaluate $\int_C \vec{F} \cdot d\vec{r}$ explicitly as a line integral, where $\vec{F} = (2-x^2)\hat{j}$ and C consists of the three line segments from $(0,0)$ to $(1,0)$ to $(0,1)$ and back to $(0,0)$.

We selected this task because there were a number of ways that the line integral could be evaluated. We believed that it was likely that the method a given student used to solve the problem would reveal his or her understanding of the problem. Only three students obtained the correct solution — one from each group. However, we were more concerned with the method each used and his or her understanding of the concept of line integral.

In the traditional group, three of the four students began by parameterizing the curve in terms of t and, with varying success, tried to evaluate integrals for each line segment in the path. One of these students, for whom this method was successful, went on to apply Stokes' Theorem to verify her answer. The fourth student drew a triangle and labeled it, but was not able to get beyond that point. Interestingly enough, this was the only student who seemed to have more than a vague idea of the meaning of the line integral. In fact, when asked about the meaning of the line integral, the student who was able to get a correct answer by doing the problem in two ways said, "... it means exactly nothing to me other than that [the answer obtained from the first method] matches [the second one]. I mean, ... math classes aren't very integrated to teach you like ... there's not that many story problems anymore."

Each of the four students in the second group started by drawing the triangle — two of them also sketched the vector field. Three of the four began by reasoning about \vec{r} and $d\vec{r}$. One student was able to carry these ideas to a successful conclusion, and a second student seemed to lose one of his integrals along the way, but would otherwise have been successful. The fourth student tried unsuccessfully to apply Stokes' Theorem. When asked to explain the idea behind line integrals, one student said they were "how much you're going with the vector field; how much of the vector field you're experiencing going from point a to point b ." Another said it was the work being done going from one point to another.

In some ways, the final group was not comparable to the other two groups, since their course was team taught by the authors in a much smaller class, and was five weeks longer (15 weeks vs. 10 weeks), although the extra time was used to cover preliminary material (multivariable calculus) which was a prerequisite for the other courses. However, looking at the second interviews of this group does give us some valuable information. As a group, these students seemed to reason about the line integral task in a more visual way. Four of the seven students, talking about the vector field in relation to the curve reasoned without computation that the integral on the segment from $(0,0)$ to $(1,0)$ would be zero. And two of these students said the integral along the $(1,0)$ to $(0,1)$ segment would be positive and the integral along the $(0,1)$ to $(0,0)$ segment would be negative, again without doing any computations. All but two students started by saying either that $\vec{r} = x\hat{i} + y\hat{j}$ or that $d\vec{r} = dx\hat{i} + dy\hat{j}$. Again, only one of those students carried the task through to a successful end, but only one of the seven seemed completely unable to handle the task.

During the second interview, students were also asked to comment on the course in order to determine their attitude towards the new materials. All four of the students in the second

group were quite enthusiastic about the labs. One student said, “[These were] probably the best labs of any math class I’ve taken here.” Another said, “... the lab activities and working with others, extremely helpful I absolutely loved the labs ... they were great.” In addition, all four students felt that the labs related well to the subject matter in the lectures and helped them understand that material more deeply. All seven of the students in the final group had positive things to say about the lab activities, however two of the students were not terribly enthusiastic — one of them preferring to work alone and the other raising issues related to group dynamics.

We also conducted informal interviews with the two instructors at the Oregon State University site and with the graduate teaching assistant who led the labs. All were enthusiastic about the labs and thought they added to the understanding and interest of their students. One of the instructors said that emphasizing the geometric meaning of vector calculus (both in the labs and in the lectures) makes it much more possible for students to make sense out of what they were doing.

5 Conclusion

There are several conclusions that one can draw from this study. First of all, when comparing these groups of students as a whole, those using Bridge Project materials did appear to reason in a more visual way. We define *geometric reasoning* broadly to mean thinking about and reasoning from geometric objects, including, but not limited to, graphs of functions and/or equations. Students using Bridge Project materials were more likely to use such geometric arguments.

Second, the enthusiasm that students had for the lab activities seems important. Students are more likely to engage in mathematics if they enjoy the work and if they find it applicable in their lives. It is clear that students in the traditional recitation section were not having a comparable experience.

Finally, the preparation of instructors to use these materials is important. The use of cooperative groups as well as the innovative way of looking at the mathematics of vector calculus may both be new ideas to faculty, and how well they are prepared to handle these new ideas will be important. Furthermore, there is some indication that it may be important to integrate more completely the ideas and thinking from the labs into the lecture sections. For instance, one of the instructors remarked that, given a second opportunity, he would emphasize the notion of differentials more in his lectures and examples.

In the spirit of Krutetskii [13], we use *harmonic reasoning* to mean the use of a combination of analytic, geometric, and numeric representations. If mathematicians and physicists speak different languages, characterized by algebra and geometry respectively, then harmonic reasoning requires the student to be bilingual. We believe the Bridge Project provides a crucial step on the road to achieving fluency in both languages.

Acknowledgments

Sections 2 and 3 are adapted from our previously published work [9]. This work was supported in part by NSF grants DUE-0088901, DUE-0231032, and DUE-0618877.

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